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A

SHORT TABLE OF INTEGRALS

BY

B. O. PEIRCE
HOLLIS PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY IN
HARVARD UNIVERSITY

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TABLE OF INTEGRALS.

I. FUNDAMENTAL FORMS.

1. $\int a\,dx = ax.$

2. $\int af(x)\,dx = a\int f(x)\,dx.$

3. $\int \frac{dx}{x} = \log x.$

4. $\int x^m\,dx = \frac{x^{m+1}}{m+1}$, when m is different from -1.

5. $\int e^x\,dx = e^x.$

6. $\int a^x \log a\,dx = a^x.$

7. $\int \frac{dx}{1+x^2} = \tan^{-1}x$, or $-\operatorname{ctn}^{-1}x.$

8. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x$, or $-\cos^{-1}x.$

9. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x$, or $-\csc^{-1}x.$

10. $\int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{versin}^{-1}x$, or $-\operatorname{coversin}^{-1}x.$

11. $\int \cos x\,dx = \sin x$, or $-$ coversin x.

12. $\int \sin x\,dx = -\cos x$, or versin x.

13. $\int \operatorname{ctn} x\,dx = \log \sin x$.

14. $\int \tan x\,dx = -\log \cos x$.

15. $\int \tan x \sec x\,dx = \sec x$.

16. $\int \sec^2 x\,dx = \tan x$.

17. $\int \csc^2 x\,dx = -\operatorname{ctn} x$.

In the following formulas, u, v, w, and y represent any functions of x:

18. $\int (u + v + w + \text{etc.})\,dx = \int u\,dx + \int v\,dx + \int w\,dx + \text{etc.}$

19 *a.* $\int u\,dv = uv - \int v\,du$.

19 *b.* $\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$.

20. $\int f(y)\,dx = \int \frac{f(y)\,dy}{\frac{dy}{dx}}$.

II. RATIONAL ALGEBRAIC FUNCTIONS.

A. — Expressions Involving $(a + bx)$.

The substitution of y or z for x, where $y \equiv a + bx$, $z \equiv (a + bx)/x$, gives

21. $$\int (a + bx)^m dx = \frac{1}{b}\int y^m dy.$$

22. $$\int x(a + bx)^m dx = \frac{1}{b^2}\int y^m (y - a)\, dy.$$

23. $$\int x^n (a + bx)^m dx = \frac{1}{b^{n+1}}\int y^m (y - a)^n dy.$$

24. $$\int \frac{x^n dx}{(a + bx)^m} = \frac{1}{b^{n+1}}\int \frac{(y - a)^n dy}{y^m}.$$

25. $$\int \frac{dx}{x^n (a + bx)^m} = -\frac{1}{a^{m+n-1}}\int \frac{(z - b)^{m+n-2} dz}{z^m}.$$

Whence

26. $$\int \frac{dx}{a + bx} = \frac{1}{b}\log(a + bx).$$

27. $$\int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)}.$$

28. $$\int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}.$$

29. $$\int \frac{x\,dx}{a + bx} = \frac{1}{b^2}[a + bx - a\log(a + bx)].$$

30. $$\int \frac{x\,dx}{(a + bx)^2} = \frac{1}{b^2}\left[\log(a + bx) + \frac{a}{a + bx}\right].$$

31. $$\int \frac{x\,dx}{(a+bx)^3} = \frac{1}{b^2}\left[-\frac{1}{a+bx} + \frac{a}{2(a+bx)^2}\right].$$

32. $$\int \frac{x^2\,dx}{a+bx} = \frac{1}{b^3}\left[\tfrac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\log(a+bx)\right].$$

33. $$\int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left[a+bx-2a\log(a+bx)-\frac{a^2}{a+bx}\right].$$

34. $$\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x}.$$

35. $$\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x}.$$

36. $$\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x}.$$

37. $$\int (a+bx)^n(a'+b'x)^m\,dx = \frac{1}{(m+n+1)b}\Bigg((a+bx)^{n+1}(a'+b'x)^m$$
$$-m(ab'-a'b)\int(a+bx)^n(a'+b'x)^{m-1}dx\Bigg).$$

38. $$\int\frac{(a+bx)^n\,dx}{(a'+b'x)^m} = -\frac{1}{(m-1)(ab'-a'b)}\Bigg(\frac{(a+bx)^{n+1}}{(a'+b'x)^{m-1}}$$
$$+(m-n-2)b\int\frac{(a+bx)^n\,dx}{(a'+b'x)^{m-1}}\Bigg)$$
$$= -\frac{1}{(m-n-1)b'}\Bigg(\frac{(a+bx)^n}{(a'+b'x)^{m-1}}$$
$$+n(ab'-a'b)\int\frac{(a+bx)^{n-1}\,dx}{(a'+b'x)^m}\Bigg)$$
$$= -\frac{1}{(m-1)b'}\Bigg(\frac{(a+bx)^n}{(a'+b'x)^{m-1}} - nb\int\frac{(a+bx)^{n-1}\,dx}{(a'+b'x)^{m-1}}\Bigg).$$

39. $$\int\frac{dx}{(a+bx)(a'+b'x)}=\frac{1}{ab'-a'b}\cdot\log\frac{a'+b'x}{a+bx}.$$

40. $$\int\frac{dx}{(a+bx)^n(a'+b'x)^m}$$
$$=\frac{1}{(m-1)(ab'-a'b)}\left(\frac{1}{(a+bx)^{n-1}(a'+b'x)^{m-1}}\right.$$
$$\left.-(m+n-2)\,b\int\frac{dx}{(a+bx)^n(a'+b'x)^{m-1}}\right).$$

41. $$\int\frac{x\,dx}{(a+bx)(a'+b'x)}$$
$$=\frac{1}{ab'-a'b}\left(\frac{a}{b}\log(a+bx)-\frac{a'}{b'}\log(a'+b'x)\right).$$

42. $$\int\frac{dx}{(a+bx)^2(a'+b'x)}$$
$$=\frac{1}{ab'-a'b}\left(\frac{1}{a+bx}+\frac{b'}{ab'-a'b}\log\frac{a'+b'x}{a+bx}\right).$$

43. $$\int\frac{x\,dx}{(a+bx)^2(a'+b'x)}$$
$$=\frac{-a}{b(ab'-a'b)(a+bx)}-\frac{a'}{(ab'-a'b)^2}\log\frac{a'+b'x}{a+bx}.$$

44. $$\int\frac{x^2dx}{(a+bx)^2(a'+b'x)}=\frac{a^2}{b^2(ab'-a'b)(a+bx)}$$
$$+\frac{1}{(ab'-a'b)^2}\left[\frac{a'^2}{b'}\log(a'+b'x)+\frac{a(ab'-2a'b)}{b^2}\log(a+bx)\right].$$

45. $$\int(a+bx)^{\frac{1}{n}}dx=\frac{n}{(n+1)b}(a+bx)^{\frac{n+1}{n}}.$$

46. $$\int\frac{dx}{(a+bx)^{\frac{1}{n}}}=\frac{n}{(n-1)b}(a+bx)^{\frac{n-1}{n}}.$$

B. — EXPRESSIONS INVOLVING $(a + bx^n)$.

47. $\int \frac{dx}{c^2 + x^2} = \frac{1}{c} \tan^{-1} \frac{x}{c}$.

48. $\int \frac{dx}{c^2 - x^2} = \frac{1}{2c} \log \frac{c + x}{c - x}$.

49. $\int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1} \left(x \sqrt{\frac{b}{a}} \right)$, if $a > 0$, $b > 0$.

50. $\int \frac{dx}{a + bx^2} = \frac{1}{2\sqrt{-ab}} \log \frac{\sqrt{a} + x\sqrt{-b}}{\sqrt{a} - x\sqrt{-b}}$, if $a > 0$, $b < 0$.

51. $\int \frac{dx}{(a + bx^2)^2} = \frac{x}{2a(a + bx^2)} + \frac{1}{2a} \int \frac{dx}{a + bx^2}$.

52. $\int \frac{dx}{(a + bx^2)^{m+1}} = \frac{1}{2ma} \frac{x}{(a + bx^2)^m} + \frac{2m - 1}{2ma} \int \frac{dx}{(a + bx^2)^m}$.

53. $\int \frac{x\,dx}{a + bx^2} = \frac{1}{2b} \log \left(x^2 + \frac{a}{b} \right)$.

54. $\int \frac{x\,dx}{(a + bx^2)^{m+1}} = \frac{1}{2} \int \frac{dz}{(a + bz)^{m+1}}$, where $z = x^2$.

55. $\int \frac{dx}{x(a + bx^2)} = \frac{1}{2a} \log \frac{x^2}{a + bx^2}$.

56. $\int \frac{x^2 dx}{a + bx^2} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a + bx^2}$.

57. $\int \frac{dx}{x^2(a + bx^2)} = -\frac{1}{ax} - \frac{b}{a} \int \frac{dx}{a + bx^2}$.

58. $\int \frac{x^2 dx}{(a + bx^2)^{m+1}} = \frac{-x}{2mb(a + bx^2)^m} + \frac{1}{2mb} \int \frac{dx}{(a + bx^2)^m}$.

59. $\int \frac{dx}{x^2(a + bx^2)^{m+1}} = \frac{1}{a} \int \frac{dx}{x^2(a + bx^2)^m} - \frac{b}{a} \int \frac{dx}{(a + bx^2)^{m+1}}$.

60. $\displaystyle\int\frac{dx}{a+bx^3}=\frac{k}{3a}\left[\tfrac{1}{2}\log\left(\frac{(k+x)^2}{k^2-kx+x^2}\right)+\sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right]$, where $bk^3=a$.

61. $\displaystyle\int\frac{x\,dx}{a+bx^3}=\frac{1}{3bk}\left[\tfrac{1}{2}\log\left(\frac{k^2-kx+x^2}{(k+x)^2}\right)+\sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right]$, where $bk^3=a$.

62. $\displaystyle\int\frac{dx}{x(a+bx^n)}=\frac{1}{an}\log\frac{x^n}{a+bx^n}.$ **63.** $\displaystyle\int\frac{dx}{(a+bx^n)^{m+1}}=\frac{1}{a}\int\frac{dx}{(a+bx^n)^m}-\frac{b}{a}\int\frac{x^n\,dx}{(a+bx^n)^{m+1}}.$

64. $\displaystyle\int\frac{x^m\,dx}{(a+bx^n)^{p+1}}=\frac{1}{b}\int\frac{x^{m-n}\,dx}{(a+bx^n)^p}-\frac{a}{b}\int\frac{x^{m-n}\,dx}{(a+bx^n)^{p+1}}.$

65. $\displaystyle\int\frac{dx}{x^m(a+bx^n)^{p+1}}=\frac{1}{a}\int\frac{dx}{x^m(a+bx^n)^p}-\frac{b}{a}\int\frac{dx}{x^{m-n}(a+bx^n)^{p+1}}.$

66. $$\int x^{m-1}(a+bx^n)^p\,dx=\begin{cases}\dfrac{1}{b(m+np)}\left[x^{m-n}(a+bx^n)^{p+1}-(m-n)a\displaystyle\int x^{m-n-1}(a+bx^n)^p\,dx\right].\\[2ex] \dfrac{1}{m+np}\left[x^m(a+bx^n)^p+npa\displaystyle\int x^{m-1}(a+bx^n)^{p-1}\,dx\right].\\[2ex] \dfrac{1}{ma}\left[x^m(a+bx^n)^{p+1}-(m+np+n)b\displaystyle\int x^{m+n-1}(a+bx^n)^p\,dx\right].\\[2ex] \dfrac{1}{an(p+1)}\left[-x^m(a+bx^n)^{p+1}+(m+np+n)\displaystyle\int x^{m-1}(a+bx^n)^{p+1}\,dx\right].\end{cases}$$

C. — Expressions Involving $(a + bx + cx^2)$.

Let $X = a + bx + cx^2$ and $q = 4ac - b^2$, then

67. $$\int \frac{dx}{X} = \frac{2}{\sqrt{q}} \tan^{-1} \frac{2cx + b}{\sqrt{q}}, \text{ when } q > 0.$$

68. $$\int \frac{dx}{X} = \frac{1}{\sqrt{-q}} \log \frac{2cx + b - \sqrt{-q}}{2cx + b + \sqrt{-q}}, \text{ when } q < 0.$$

69. $$\int \frac{dx}{X^2} = \frac{2cx + b}{qX} + \frac{2c}{q} \int \frac{dx}{X}.$$

70. $$\int \frac{dx}{X^3} = \frac{2cx + b}{q} \left(\frac{1}{2X^2} + \frac{3c}{qX} \right) + \frac{6c^2}{q^2} \int \frac{dx}{X}.$$

71. $$\int \frac{dx}{X^{n+1}} = \frac{2cx + b}{nqX^n} + \frac{2(2n - 1)c}{qn} \int \frac{dx}{X^n}.$$

72. $$\int \frac{x\,dx}{X} = \frac{1}{2c} \log X - \frac{b}{2c} \int \frac{dx}{X}.$$

73. $$\int \frac{x\,dx}{X^2} = -\frac{bx + 2a}{qX} - \frac{b}{q} \int \frac{dx}{X}.$$

74. $$\int \frac{x\,dx}{X^{n+1}} = -\frac{2a + bx}{nqX^n} - \frac{b(2n - 1)}{nq} \int \frac{dx}{X^n}.$$

75. $$\int \frac{x^2}{X}\,dx = \frac{x}{c} - \frac{b}{2c^2} \log X + \frac{b^2 - 2ac}{2c^2} \int \frac{dx}{X}.$$

76. $$\int \frac{x^2}{X^2}\,dx = \frac{(b^2 - 2ac)x + ab}{cqX} + \frac{2a}{q} \int \frac{dx}{X}.$$

77. $$\int \frac{x^m\,dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n - m + 1)cX^n} - \frac{n - m + 1}{2n - m + 1} \cdot \frac{b}{c} \int \frac{x^{m-1}\,dx}{X^{n+1}}$$
$$+ \frac{m - 1}{2n - m + 1} \cdot \frac{a}{c} \int \frac{x^{m-2}\,dx}{X^{n+1}}.$$

78. $$\int\frac{dx}{xX}=\frac{1}{2a}\log\frac{x^2}{X}-\frac{b}{2a}\int\frac{dx}{X}.$$

79. $$\int\frac{dx}{x^2X}=\frac{b}{2a^2}\log\frac{X}{x^2}-\frac{1}{ax}+\left(\frac{b^2}{2a^2}-\frac{c}{a}\right)\int\frac{dx}{X}.$$

80. $$\int\frac{dx}{x^mX^{n+1}}=-\frac{1}{(m-1)ax^{m-1}X^n}-\frac{n+m-1}{m-1}\cdot\frac{b}{a}\int\frac{dx}{x^{m-1}X^{n+1}}$$
$$-\frac{2n+m-1}{m-1}\cdot\frac{c}{a}\int\frac{dx}{x^{m-2}X^{n+1}}.$$

81. $$\int X^n\,dx=\frac{1}{2(2n+1)c}\left((b+2cx)X^n+nq\int X^{n-1}\,dx\right).$$

82. $$\int\frac{dx}{xX^n}=\frac{1}{2a(n-1)X^{n-1}}-\frac{b}{2a}\int\frac{dx}{X^n}+\frac{1}{a}\int\frac{dx}{xX^{n-1}}.$$

83. $$\int\frac{dx}{(a'+b'x)X}=\frac{1}{2(ab'^2-a'bb'+a'^2c)}\Bigg(b'(\log(a'+b'x)^2$$
$$-\log X)+(2a'c-bb')\int\frac{dx}{X}\Bigg).$$

84. $$\int(a'+b'x)X^n\,dx=\frac{b'X^{n+1}}{2(n+1)c}+\frac{2a'c-bb'}{2c}\int X^n\,dx.$$

85. $$\int\frac{(a'+b'x)\,dx}{X^n}=-\frac{b'}{2(n-1)cX^{n-1}}+\frac{2a'c-bb'}{2c}\int\frac{dx}{X^n}.$$

86. $$\int(a'+b'x)^mX^n\,dx=\frac{1}{(m+2n+1)c}\Bigg(b'(a'+b'x)^{m-1}X^{n+1}$$
$$+(m+n)(2a'c-bb')\int(a'+b'x)^{m-1}X^n\,dx$$
$$-(m-1)(ab'^2-a'bb'+ca'^2)\int(a'+b'x)^{m-2}X^n\,dx\Bigg).$$

87. $$\begin{aligned}\int\frac{(a'+b'x)^m\,dx}{X^n} &= \frac{1}{q\,(n-1)}\left(\frac{(b+2\,cx)(a'+b'x)^m}{X^{n-1}}\right.\\
&\qquad -2\,(m-2\,n+3)\,c\int\frac{(a'+b'x)^m\,dx}{X^{n-1}}\\
&\qquad \left.+\,m\,(2\,a'c-bb')\int\frac{(a'+b'x)^{m-1}\,dx}{X^{n-1}}\right)\\
&= \frac{1}{(m-2\,n+1)\,c}\left(\frac{b'\,(a'+b'x)^{m-1}}{X^{n-1}}\right.\\
&\qquad +(m-n)\,(2\,a'c-bb')\int\frac{(a'+b'x)^{m-1}\,dx}{X^n}\\
&\qquad \left.-\,(m-1)\,(ab'^2-a'bb'+ca'^2)\int\frac{(a'+b'x)^{m-2}\,dx}{X^n}\right).\end{aligned}$$

88. $$\begin{aligned}\int\frac{X^n\,dx}{(a'+b'x)^m}\\
&= \frac{1}{b'^2\,(m-1)}\left(\frac{-\,b'X^n}{(a'+b'x)^{m-1}}\right.\\
&\qquad +n\,(bb'-2\,a'c)\int\frac{X^{n-1}\,dx}{(a'+b'x)^{m-1}}\\
&\qquad \left.+\,2\,nc\int\frac{X^{n-1}\,dx}{(a'+b'x)^{m-2}}\right)\\
&= -\frac{1}{(m-2\,n-1)\,b'^2}\left(\frac{+\,b'X^n}{(a'+b'x)^{m-1}}\right.\\
&\qquad +2\,b'n\,(ab'^2-a'bb'+ca'^2)\int\frac{X^{n-1}\,dx}{(a'+b'x)^m}\\
&\qquad \left.+\,n\,(bb'-2\,a'c)\int\frac{X^{n-1}\,dx}{(a'+b'x)^{m-1}}\right).\end{aligned}$$

89. $$\int\frac{dx}{(a'+b'x)^m X^n}$$

$$= -\frac{1}{(m-1)(ab'^2-a'bb'+ca'^2)}\left(\frac{b'}{(a'+b'x)^{m-1}X^{n-1}}\right.$$

$$+(m+n-2)(bb'-2ca')\int\frac{dx}{(a'+b'x)^{m-1}X^n}$$

$$\left.+(m+2n-3)c\int\frac{dx}{(a'+b'x)^{m-2}X^n}\right)$$

$$=\frac{1}{2(ab'^2-a'bb'+ca'^2)}\left(\frac{b'}{(n-1)(a'+b'x)^{m-1}X^{n-1}}\right.$$

$$+(2a'c-bb')\int\frac{dx}{(a'+b'x)^{m-1}X^n}$$

$$\left.+\frac{(m+2n-3)b'^2}{n+1}\int\frac{dx}{(a'+b'x)^m X^{n-1}}\right).$$

If $ab'^2-a'bb'+ca'^2=0$,

$$\int\frac{dx}{(a'+b'x)^m X^n}$$

$$=\frac{-1}{(m+n-1)(bb'-2a'c)}\left(\frac{b'}{(a'+b'x)^m X^{n-1}}\right.$$

$$\left.+(m+2n-2)c\int\frac{dx}{(a'+b'x)^{m-1}X^n}\right).$$

D. — Rational Fractions.

Every proper fraction can be represented by the general form:

$$\frac{f(x)}{F(x)}=\frac{g_1x^{n-1}+g_2x^{n-2}+g_3x^{n-3}+\cdots+g_n}{x^n+k_1x^{n-1}+k_2x^{n-2}+\cdots+k_n}.$$

If a, b, c, etc., are the roots of the equation $F(x)=0$, so that

$$F(x)=(x-a)^p(x-b)^q(x-c)^r\cdots,$$

then

$$\begin{aligned}\frac{f(x)}{F(x)} = {} & \frac{A_1}{(x-a)^p} + \frac{A_2}{(x-a)^{p-1}} + \frac{A_3}{(x-a)^{p-2}} + \cdots + \frac{A_p}{x-a} \\ & + \frac{B_1}{(x-b)^q} + \frac{B_2}{(x-b)^{q-1}} + \frac{B_3}{(x-b)^{q-2}} + \cdots + \frac{B^q}{x-b} \\ & + \frac{C_1}{(x-c)^r} + \frac{C_2}{(x-c)^{r-1}} + \frac{C_3}{(x-c)^{r-2}} + \cdots + \frac{C_r}{x-c} \\ & + \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots,\end{aligned}$$

where the numerators of the separate fractions may be determined by the equations

$$A_m = \frac{\phi_1^{[m-1]}(a)}{(m-1)!}, \quad B_m = \frac{\phi_2^{[m-1]}(b)}{(m-1)!} \quad \text{etc., etc.}$$

$$\phi_1(x) = \frac{f(x)\,(x-a)^p}{F(x)}, \quad \phi_2(x) = \frac{f(x)\,(x-b)^q}{F(x)}, \quad \text{etc., etc.}$$

If a, b, c, etc., are single roots, then $p = q = r = \cdots = 1$, and

$$\frac{f(x)}{F(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \cdots$$

where $$A = \frac{f(a)}{F'(a)}, \quad B = \frac{f(b)}{F'(b)}, \quad \text{etc.}$$

The simpler fractions, into which the original fraction is thus divided, may be integrated by means of the formulas:

90. $$\int \frac{h\,dx}{(mx+n)^l} = \int \frac{h\,d(mx+n)}{m\,(mx+n)^l} = \frac{h}{m\,(1-l)\,(mx+n)^{l-1}},$$

and $$\int \frac{h\,dx}{mx+n} = \frac{h}{m} \log\,(mx+n).$$

If any of the roots of the equation $f(x) = 0$ are imaginary, the parts of the integral which arise from conjugate roots can be combined and the integral brought into a real form. The following formula, in which $i = \sqrt{-1}$, is often useful in combining logarithms of conjugate complex quantities:

$$\log(x \pm yi) = \tfrac{1}{2}\log(x^2 + y^2) \pm i \tan^{-1}\frac{y}{x}.$$

The identities given below are sometimes convenient:

$$\frac{1}{(a + bx^2)(a' + b'x^2)} \equiv \frac{1}{a'b - ab'} \cdot \left[\frac{b}{a + bx^2} - \frac{b'}{a' + b'x^2}\right],$$

$$\frac{m + nx}{(k + lx)(a + bx + cx^2)} \equiv \frac{1}{al^2 + ck^2 - bkl} \cdot$$

$$\left[\frac{l(ml - nk)}{k + lx} + \frac{c(nk - ml)x + (aln + ckm - blm)}{a + bx + cx^2}\right],$$

$$\frac{l + mx^n}{(a + bx^n)(a' + b'x^n)} \equiv \frac{1}{a'b - ab'} \cdot \left[\frac{bl - am}{a + bx^n} + \frac{a'm - b'l}{a' + b'x^n}\right].$$

III. IRRATIONAL ALGEBRAIC FUNCTIONS.

A. — EXPRESSIONS INVOLVING $\sqrt{a+bx}$.

The substitution of a new variable of integration, $y=\sqrt{a+bx}$, gives

91. $$\int\sqrt{a+bx}\,dx=\frac{2}{3\,b}\sqrt{(a+bx)^3}.$$

92. $$\int x\sqrt{a+bx}\,dx=-\frac{2\,(2\,a-3\,bx)\sqrt{(a+bx)^3}}{15\,b^2}.$$

93. $$\int x^2\sqrt{a+bx}\,dx=\frac{2\,(8\,a^2-12\,abx+15\,b^2x^2)\sqrt{(a+bx)^3}}{105\,b^3}.$$

94. $$\int\frac{\sqrt{a+bx}}{x}\,dx=2\sqrt{a+bx}+a\int\frac{dx}{x\sqrt{a+bx}}.$$

95. $$\int\frac{dx}{\sqrt{a+bx}}=\frac{2\sqrt{a+bx}}{b}.$$

96. $$\int\frac{x\,dx}{\sqrt{a+bx}}=-\frac{2\,(2\,a-bx)}{3\,b^2}\sqrt{a+bx}.$$

97. $$\int\frac{x^2\,dx}{\sqrt{a+bx}}=\frac{2\,(8\,a^2-4\,abx+3\,b^2x^2)}{15\,b^3}\sqrt{a+bx}.$$

98. $$\int\frac{dx}{x\sqrt{a+bx}}=\frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right),\text{ for }a>0.$$

99. $$\int\frac{dx}{x\sqrt{a+bx}}=\frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}},\text{ for }a<0.$$

100. $$\int \frac{dx}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{a+bx}}.$$

101. $$\int (a+bx)^{\pm\frac{n}{2}}\,dx = \frac{2}{b}\int y^{1\pm n}\,dy = \frac{2(a+bx)^{\frac{2\pm n}{2}}}{b(2\pm n)}.$$

102. $$\int x(a+bx)^{\pm\frac{n}{2}}dx = \frac{2}{b^2}\left[\frac{(a+bx)^{\frac{4\pm n}{2}}}{4\pm n} - \frac{a(a+bx)^{\frac{2\pm n}{2}}}{2\pm n}\right].$$

103. $$\int \frac{x^m\,dx}{\sqrt{a+bx}} = \frac{2x^m\sqrt{a+bx}}{(2m+1)b} - \frac{2ma}{(2m+1)b}\int \frac{x^{m-1}\,dx}{\sqrt{a+bx}}.$$

104. $$\int \frac{dx}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{(2n-2)a}\int \frac{dx}{x^{n-1}\sqrt{a+bx}}.$$

105. $$\int \frac{(a+bx)^{\frac{n}{2}}dx}{x} = b\int (a+bx)^{\frac{n-2}{2}}dx + a\int \frac{(a+bx)^{\frac{n-2}{2}}}{x}\,dx.$$

106. $$\int \frac{dx}{x(a+bx)^{\frac{m}{2}}} = \frac{1}{a}\int \frac{dx}{x(a+bx)^{\frac{m-2}{2}}} - \frac{b}{a}\int \frac{dx}{(a+bx)^{\frac{m}{2}}}.$$

107. $$\int f(x, \sqrt[n]{a+bx})dx = \frac{n}{b}\int f\left(\frac{z^n-a}{b}, z\right)z^{n-1}dz,$$

where $z^n = a + bx$.

108. $$\int (a+bx)^{\frac{m}{n}}dx = \frac{n(a+bx)^{\frac{m+n}{n}}}{b(m+n)}.$$

109. $$\int f(x, (a+bx)^{\frac{m}{n}}, (a+bx)^{\frac{p}{q}}, \cdots)\,dx$$
$$= \frac{s}{b}\int f\left(\frac{y^s-a}{b}, y^{\frac{ms}{n}}, y^{\frac{ps}{q}}, \cdots\right)y^{s-1}dy,$$

where $y^s = a + bx$, and s is the least common multiple of n, q, etc.

B.—Expressions Involving Both $\sqrt{a+bx}$ and $\sqrt{a'+b'x}$.

Let $u = a + bx$, $v = a' + b'x$, and $k = ab' - a'b$, then

110. $$\int \sqrt{uv}\,dx = \frac{k+2bv}{4bb'}\sqrt{uv} - \frac{k^2}{8bb'}\int\frac{dx}{\sqrt{uv}}.$$

111. $$\int\frac{\sqrt{v}\,dx}{\sqrt{u}} = \frac{1}{b}\sqrt{uv} - \frac{k}{2b}\int\frac{dx}{\sqrt{uv}}.$$

112. $$\int\frac{x\,dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{bb'} - \frac{ab'+a'b}{2bb'}\int\frac{dx}{\sqrt{uv}}.$$

113. $$\int\frac{dx}{\sqrt{uv}} = \frac{2}{\sqrt{bb'}}\log\left(\sqrt{bb'u} + b\sqrt{v}\right)$$
$$= \frac{2}{\sqrt{-bb'}}\tan^{-1}\frac{\sqrt{-bb'u}}{b\sqrt{v}}$$
$$= -\frac{1}{\sqrt{-bb'}}\sin^{-1}\frac{2bb'x+a'b+ab'}{k}.$$

114. $$\int\frac{dx}{v\sqrt{u}} = \frac{1}{\sqrt{kb'}}\log\frac{b'\sqrt{u}-\sqrt{kb'}}{b'\sqrt{u}+\sqrt{kb'}} = \frac{2}{\sqrt{-kb'}}\tan^{-1}\frac{b'\sqrt{u}}{\sqrt{-kb'}}.$$

115. $$\int\frac{dx}{v\sqrt{uv}} = -\frac{2\sqrt{u}}{k\sqrt{v}}.$$

116. $$\int v^m\sqrt{u}\,dx = \frac{1}{(2m+3)b'}\left(2v^{m+1}\sqrt{u} + k\int\frac{v^m\,dx}{\sqrt{u}}\right).$$

117. $$\int\frac{\sqrt{u}\,dx}{v^m} = -\frac{1}{(2m-3)b'}\left(\frac{2\sqrt{u}}{v^{m-1}} + k\int\frac{dx}{v^m\sqrt{u}}\right)$$
$$= \frac{1}{(m-1)b'}\left(-\frac{\sqrt{u}}{v^{m-1}} + \tfrac{1}{2}b\int\frac{dx}{v^{m-1}\sqrt{u}}\right).$$

118. $$\int\frac{v^m\,dx}{\sqrt{u}} = \frac{2}{(2m+1)b}\left(v^m\sqrt{u} - mk\int\frac{v^{m-1}\,dx}{\sqrt{u}}\right).$$

119. $$\int \frac{dx}{v^m\sqrt{u}} = -\frac{1}{(m-1)k}\left(\frac{\sqrt{u}}{v^{m-1}} + (m-\tfrac{3}{2})b\int\frac{dx}{v^{m-1}\sqrt{u}}\right).$$

120. $$\begin{aligned}\int v^m u^{n-\frac{1}{2}}dx = \frac{1}{(2m+2n+1)b'}\Big(&2v^{m+1}u^{n-\frac{1}{2}}\\ &+(2n-1)k\int v^m u^{n-\frac{3}{2}}dx\Big).\end{aligned}$$

121. $$\begin{aligned}\int v^m u^{-(n+\frac{1}{2})}dx &= \frac{1}{(2n-1)k}\Big(2v^{m+1}u^{-(n-\frac{1}{2})}\\ &\quad -(2m-2n+3)b'\int v^m u^{-(n-\frac{1}{2})}dx\Big)\\ &= \frac{2}{(2n-1)b}\Big(-v^m u^{-(n-\frac{1}{2})}\\ &\quad + mb'\int v^{m-1}u^{-(n-\frac{1}{2})}dx\Big).\end{aligned}$$

122. $$\begin{aligned}\int v^{-m}u^{(n-\frac{1}{2})}dx &= \frac{-1}{(2m-2n-1)b'}\Big(2u^{n-\frac{1}{2}}v^{-(m-1)}\\ &\quad +(2n-1)k\int u^{n-\frac{3}{2}}v^{-m}dx\Big)\\ &= \frac{1}{(m-1)b'}\Big(-u^{n-\frac{1}{2}}v^{-(m-1)}\\ &\quad +(n-\tfrac{1}{2})b\int u^{n-\frac{3}{2}}v^{-(m-1)}dx\Big).\end{aligned}$$

123. $$\begin{aligned}\int v^{-m}u^{-(n+\frac{1}{2})}dx = \frac{1}{(2n-1)k}\Big(&2v^{-(m-1)}u^{-(n-\frac{1}{2})}\\ &+(2m+2n-3)b'\int v^{-m}u^{-(n-\frac{1}{2})}dx\Big).\end{aligned}$$

C. — EXPRESSIONS* INVOLVING $\sqrt{x^2 \pm a^2}$ AND $\sqrt{a^2 - x^2}$.

124. $$\int \sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2 \log\left(x + \sqrt{x^2 \pm a^2}\right)\right].$$

125. $$\int \sqrt{a^2 - x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right).$$

126. $$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left(x + \sqrt{x^2 \pm a^2}\right).$$

127. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}, \text{ or } -\cos^{-1}\frac{x}{a}.$$

128. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\cos^{-1}\frac{a}{x}, \text{ or } \frac{1}{a}\sec^{-1}\frac{x}{a}.$$

129. $$\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 \pm x^2}}{x}\right).$$

130. $$\int \frac{\sqrt{a^2 \pm x^2}}{x}\,dx = \sqrt{a^2 \pm x^2} - a \log\frac{a + \sqrt{a^2 \pm x^2}}{x}.$$

131. $$\int \frac{\sqrt{x^2 - a^2}}{x}\,dx = \sqrt{x^2 - a^2} - a\cos^{-1}\frac{a}{x}.$$

132. $$\int \frac{x\,dx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2}.$$

133. $$\int \frac{x\,dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}.$$

134. $$\int x\sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{3}\sqrt{(x^2 \pm a^2)^3}.$$

135. $$\int x\sqrt{a^2 - x^2}\,dx = -\tfrac{1}{3}\sqrt{(a^2 - x^2)^3}.$$

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* These equations are all special cases of more general equations given in the next section. For additional formulas consult Equation 66.

136. $\int \sqrt{(x^2 \pm a^2)^3}\,dx$

$$= \tfrac{1}{4}\left[x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2x}{2}\sqrt{x^2 \pm a^2} + \frac{3a^4}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right)\right].$$

137. $\int \sqrt{(a^2 - x^2)^3}\,dx$

$$= \tfrac{1}{4}\left[x\sqrt{(a^2 - x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2 - x^2} + \frac{3a^4}{2}\sin^{-1}\frac{x}{a}\right].$$

138. $\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}.$

139. $\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2\sqrt{a^2 - x^2}}.$

140. $\int \frac{x\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}.$

141. $\int \frac{x\,dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}}.$

142. $\int x\sqrt{(x^2 \pm a^2)^3}\,dx = \tfrac{1}{5}\sqrt{(x^2 \pm a^2)^5}.$

143. $\int x\sqrt{(a^2 - x^2)^3}\,dx = -\tfrac{1}{5}\sqrt{(a^2 - x^2)^5}.$

144. $\int x^2\sqrt{x^2 \pm a^2}\,dx$

$$= \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\log\left(x + \sqrt{x^2 \pm a^2}\right).$$

145. $\int x^2\sqrt{a^2 - x^2}\,dx$

$$= -\frac{x}{4}\sqrt{(a^2 - x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{a}\right).$$

146. $$\int \frac{\sqrt{a^2 \pm x^2}\,dx}{x^3} = -\frac{\sqrt{a^2 \pm x^2}}{2\,x^2} \pm \frac{1}{2}\int \frac{dx}{x\sqrt{a^2 \pm x^2}}.$$

147. $$\int x^3\sqrt{a^2 \pm x^2}\,dx = (\pm \tfrac{1}{5}\,x^2 - \tfrac{2}{15}\,a^2)\sqrt{(a^2 \pm x^2)^3}.$$

148. $$\int \frac{dx}{x^3\sqrt{a^2 \pm x^2}} = -\frac{\sqrt{a^2 \pm x^2}}{2\,a^2x^2} \mp \frac{1}{2\,a^2}\int \frac{dx}{x\sqrt{a^2 \pm x^2}}.$$

149. $$\int \frac{dx}{x^3\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{2\,a^2x^2} + \frac{1}{2\,a^3}\sec^{-1}\left(\frac{x}{a}\right).$$

150. $$\int \frac{x^2dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\log\,(x + \sqrt{x^2 \pm a^2}).$$

151. $$\int \frac{x^2dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

152. $$\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{a^2x}.$$

153. $$\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2x}.$$

154. $$\int \frac{\sqrt{x^2 \pm a^2}\,dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log\,(x + \sqrt{x^2 \pm a^2}).$$

155. $$\int \frac{\sqrt{a^2 - x^2}}{x^2}\,dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{a}.$$

156. $$\int \frac{x^2dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log\,(x + \sqrt{x^2 \pm a^2}).$$

157. $$\int \frac{x^2dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1}\frac{x}{a}.$$

158. $$\int \frac{f(x^2)\,dx}{\sqrt{a+cx^2}} = g\int f\left(\frac{au^2}{g^2-cu^2}\right)\frac{du}{(g^2-cu^2)},$$

where $u = \frac{gx}{\sqrt{a+cx^2}}$.

159. $$\int \frac{xf(x^2)\,dx}{\sqrt{a+cx^2}} = \frac{1}{c}\int f\left(\frac{u^2-a}{c}\right)du, \text{ where } u^2 = a + cx^2.$$

D. — EXPRESSIONS INVOLVING $\sqrt{a+bx+cx^2}$.

Let $X = a + bx + cx^2$, $q = 4ac - b^2$, and $k = \frac{4c}{q}$. In order to rationalize the function $f(x, \sqrt{a+bx+cx^2})$ we may put $\sqrt{a+bx+cx^2} = \sqrt{\pm c}\sqrt{A+Bx\pm x^2}$, according as c is positive or negative, and then substitute for x a new variable z, such that

$$z = \sqrt{A+Bx+x^2} \pm x, \text{ if } c > 0.$$

$$z = \frac{\sqrt{A+Bx-x^2}-\sqrt{A}}{x}, \text{ if } c < 0 \text{ and } \frac{a}{-c} > 0.$$

$z = \sqrt{\frac{x-\beta}{\alpha-x}}$, where α and β are the roots of the equation

$$A + Bx - x^2 = 0, \text{ if } c < 0 \text{ and } \frac{a}{-c} < 0.$$

By rationalization, or by the aid of reduction formulas, may be obtained the values of the following integrals:

160. $$\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}}\log\left(\sqrt{X} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right), \text{ if } c > 0.$$

161. $$\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{-2cx-b}{\sqrt{b^2-4ac}}\right), \text{ if } c < 0.$$

162. $$\int \frac{dx}{X\sqrt{X}} = \frac{2(2cx+b)}{q\sqrt{X}}.$$

163. $$\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx+b)}{3q\sqrt{X}}\left(\frac{1}{X}+2k\right).$$

164. $$\int \frac{dx}{X^n\sqrt{X}} = \frac{2(2cx+b)\sqrt{X}}{(2n-1)qX^n} + \frac{2k(n-1)}{2n-1}\int \frac{dx}{X^{n-1}\sqrt{X}}.$$

165. $$\int \sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{4c} + \frac{1}{2k}\int \frac{dx}{\sqrt{X}}.$$

166. $$\int X\sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{8c}\left(X+\frac{3}{2k}\right) + \frac{3}{8k^2}\int \frac{dx}{\sqrt{X}}.$$

167. $$\int X^2\sqrt{X}\,dx$$
$$= \frac{(2cx+b)\sqrt{X}}{12c}\left(X^2+\frac{5X}{4k}+\frac{15}{8k^2}\right) + \frac{5}{16k^3}\int \frac{dx}{\sqrt{X}}.$$

168. $$\int X^n\sqrt{X}\,dx = \frac{(2cx+b)X^n\sqrt{X}}{4(n+1)c} + \frac{2n+1}{2(n+1)k}\int \frac{X^n\,dx}{\sqrt{X}}.$$

169. $$\int \frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c}\int \frac{dx}{\sqrt{X}}.$$

170. $$\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}}.$$

171. $$\int \frac{x\,dx}{X^n\sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)cX^n} - \frac{b}{2c}\int \frac{dx}{X^n\sqrt{X}}.$$

172. $$\int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2c}-\frac{3b}{4c^2}\right)\sqrt{X} + \frac{3b^2-4ac}{8c^2}\int \frac{dx}{\sqrt{X}}.$$

173. $$\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{cq\sqrt{X}} + \frac{1}{c}\int \frac{dx}{\sqrt{X}}.$$

174. $$\int \frac{x^2\,dx}{X^n\sqrt{X}} = \frac{(2\,b^2 - 4\,ac)\,x + 2\,ab}{(2\,n-1)\,cq\,X^{n-1}\sqrt{X}} + \frac{4\,ac + (2\,n-3)\,b^2}{(2\,n-1)\,cq}\int\frac{dx}{X^{n-1}\sqrt{X}}.$$

175. $$\int\frac{x^3\,dx}{\sqrt{X}} = \left(\frac{x^2}{3\,c} - \frac{5\,bx}{12\,c^2} + \frac{5\,b^2}{8\,c^3} - \frac{2\,a}{3\,c^2}\right)\sqrt{X} + \left(\frac{3\,ab}{4\,c^2} - \frac{5\,b^3}{16\,c^3}\right)\int\frac{dx}{\sqrt{X}}.$$

176. $$\int x\sqrt{X}\,dx = \frac{X\sqrt{X}}{3\,c} - \frac{b}{2\,c}\int\sqrt{X}\,dx.$$

177. $$\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5\,c} - \frac{b}{2\,c}\int X\sqrt{X}\,dx.$$

178. $$\int\frac{xX^n\,dx}{\sqrt{X}} = \frac{X^n\sqrt{X}}{(2\,n+1)\,c} - \frac{b}{2\,c}\int\frac{X^n\,dx}{\sqrt{X}}.$$

179. $$\int x^2\sqrt{X}\,dx = \left(x - \frac{5\,b}{6\,c}\right)\frac{X\sqrt{X}}{4\,c} + \frac{5\,b^2 - 4\,ac}{16\,c^2}\int\sqrt{X}\,dx.$$

180. $$\int\frac{x^2X^n\,dx}{\sqrt{X}} = \frac{xX^n\sqrt{X}}{2\,(n+1)\,c} - \frac{(2\,n+3)\,b}{4\,(n+1)\,c}\int\frac{xX^n\,dx}{\sqrt{X}} - \frac{a}{2\,(n+1)\,c}\int\frac{X^n\,dx}{\sqrt{X}}.$$

181. $$\int x^3\sqrt{X}\,dx = \left(x^2 - \frac{7\,bx}{8\,c} + \frac{35\,b^2}{48\,c^2} - \frac{2\,a}{3\,c}\right)\frac{X\sqrt{X}}{5\,c} + \left(\frac{3\,ab}{8\,c^2} - \frac{7\,b^3}{32\,c^3}\right)\int\sqrt{X}\,dx.$$

182. $$\int\frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{X}+\sqrt{a}}{x} + \frac{b}{2\sqrt{a}}\right),\ \text{if } a > 0.$$

183. $$\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{bx + 2a}{x\sqrt{b^2 - 4ac}}\right), \text{ if } a < 0.$$

184. $$\int \frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx}, \text{ if } a = 0.$$

185. $$\int \frac{dx}{xX^n\sqrt{X}}$$
$$= \frac{\sqrt{X}}{(2n-1)aX^n} + \frac{1}{a}\int \frac{dx}{xX^{n-1}\sqrt{X}} - \frac{b}{2a}\int \frac{dx}{X^n\sqrt{X}}.$$

186. $$\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{X}}.$$

187. $$\int \frac{\sqrt{X}\,dx}{x} = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + a\int \frac{dx}{x\sqrt{X}}.$$

188. $$\int \frac{X^n\,dx}{x\sqrt{X}} = \frac{X^n}{(2n-1)\sqrt{X}} + a\int \frac{X^{n-1}\,dx}{x\sqrt{X}} + \frac{b}{2}\int \frac{X^{n-1}\,dx}{\sqrt{X}}.$$

189. $$\int \frac{\sqrt{X}\,dx}{x^2} = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}} + c\int \frac{dx}{\sqrt{X}}.$$

190. $$\int \frac{x^m\,dx}{X^n\sqrt{X}} = \frac{1}{c}\int \frac{x^{m-2}\,dx}{X^{n-1}\sqrt{X}} - \frac{b}{c}\int \frac{x^{m-1}\,dx}{X^n\sqrt{X}} - \frac{a}{c}\int \frac{x^{m-2}\,dx}{X^n\sqrt{X}}.$$

191. $$\int \frac{x^mX^n\,dx}{\sqrt{X}} = \frac{x^{m-1}X^n\sqrt{X}}{(2n+m)c} - \frac{(2n+2m-1)b}{2c(2n+m)}\int \frac{x^{m-1}X^n\,dx}{\sqrt{X}}$$
$$- \frac{(m-1)a}{(2n+m)c}\int \frac{x^{m-2}X^n\,dx}{\sqrt{X}}.$$

192. $$\int \frac{dx}{x^mX^n\sqrt{X}}$$
$$= -\frac{\sqrt{X}}{(m-1)ax^{m-1}X^n} - \frac{(2n+2m-3)b}{2a(m-1)}\int \frac{dx}{x^{m-1}X^n\sqrt{X}}$$
$$- \frac{(2n+m-2)c}{(m-1)a}\int \frac{dx}{x^{m-2}X^n\sqrt{X}}.$$

193. $$\int \frac{X^n\,dx}{x^m\sqrt{X}} = -\frac{X^{n-1}\sqrt{X}}{(m-1)\,x^{m-1}} + \frac{(2n-1)\,b}{2\,(m-1)}\int\frac{X^{n-1}\,dx}{x^{m-1}\sqrt{X}} + \frac{(2n-1)\,c}{m-1}\int\frac{X^{n-1}\,dx}{x^{m-2}\sqrt{X}}.$$

194. $$\int f(x,\ \sqrt{(x-a)\,(x-b)})\,dx = 2\,(a-b)\int f\left\{\frac{bu^2-a}{u^2-1},\ \frac{u\,(b-a)}{u^2-1}\right\}\frac{u\,du}{(u^2-1)^2},$$

where $u^2\,(x-b) = x-a$.

E. — Expressions Involving Products of Powers of $(a'+b'x)$ and $\sqrt{a+bx+cx^2}$.

Let $X = a+bx+cx^2$, $v = a'+b'x$, $q = 4\,ac - b^2$,
$\beta = bb' - 2\,a'c$, $k = ab'^2 - a'bb' + ca'^2$, then

195. $$\int\frac{dx}{v\sqrt{X}} = \frac{1}{\sqrt{k}}\log\frac{2\,k+\beta v - 2\,b'\sqrt{kX}}{v} = \frac{1}{\sqrt{-k}}\tan^{-1}\frac{2\,k+\beta v}{2b'\sqrt{-kX}} = \frac{1}{\sqrt{-k}}\sin^{-1}\frac{2\,k+\beta v}{b'v\sqrt{-q}},\ \text{if } k\neq 0.$$

196. $$\int\frac{dx}{v\sqrt{X}} = -\frac{2\,b'\sqrt{X}}{\beta v},\ \text{if } k=0:$$

thus, $$\int\frac{dx}{(x\pm 1)\sqrt{x^2-1}} = \pm\sqrt{\frac{x\mp 1}{x\pm 1}}.$$

197. $$\int\frac{dx}{v^2\sqrt{X}} = -\frac{b'\sqrt{X}}{kv} - \frac{\beta}{2\,k}\int\frac{dx}{v\sqrt{X}}.$$

198. $$\int\frac{dx}{v^2\sqrt{X}} = -\frac{2\,b'\sqrt{X}}{3\,\beta v^2} - \frac{2\,c}{3\,\beta}\int\frac{dx}{v\sqrt{X}},\ \text{if } k=0.$$

199. $$\int\frac{dx}{vX\sqrt{X}}=\frac{1}{k}\left(\frac{b'}{\sqrt{X}}-\tfrac{1}{2}\beta\int\frac{dx}{X\sqrt{X}}+b'^2\int\frac{dx}{v\sqrt{X}}\right).$$

200. $$\int\frac{v\,dx}{X\sqrt{X}}=-\frac{2(2k+\beta v)}{b'q\sqrt{X}}.$$

201. $$\int\frac{v\,dx}{\sqrt{X}}=\frac{b'\sqrt{X}}{c}-\frac{\beta}{2c}\int\frac{dx}{\sqrt{X}}.$$

202. $$\int v\sqrt{X}\,dx=\frac{b'X\sqrt{X}}{3c}-\frac{\beta}{2c}\int\sqrt{X}\,dx.$$

203. $$\int\frac{v\,dx}{X^n\sqrt{X}}=-\frac{b'\sqrt{X}}{(2n-1)cX^n}-\frac{\beta}{2c}\int\frac{dx}{X^n\sqrt{X}}.$$

204. $$\int\frac{vX^n\,dx}{\sqrt{X}}=\frac{b'X^n\sqrt{X}}{(2n+1)c}-\frac{\beta}{2c}\int\frac{X^n\,dx}{\sqrt{X}}.$$

205. $$\int\frac{dx}{v^m\sqrt{X}}=-\frac{b'\sqrt{X}}{(m-1)kv^{m-1}}-\frac{(2m-3)\beta}{2(m-1)k}\int\frac{dx}{v^{m-1}\sqrt{X}}$$
$$-\frac{(m-2)c}{(m-1)k}\int\frac{dx}{v^{m-2}\sqrt{X}},\ \text{if } k\neq 0.$$

206. $$\int\frac{dx}{v^m\sqrt{X}}=-\frac{2b'\sqrt{X}}{(2m-1)\beta v^m}$$
$$-\frac{2(m-1)c}{(2m-1)\beta}\int\frac{dx}{v^{m-1}\sqrt{X}},\ \text{if } k=0.$$

207. $$\int\frac{\sqrt{X}\,dx}{v^m}=-\frac{b'X\sqrt{X}}{(m-1)kv^{m-1}}-\frac{(2m-5)\beta}{2(m-1)k}\int\frac{\sqrt{X}\,dx}{v^{m-1}}$$
$$-\frac{(m-4)c}{(m-1)k}\int\frac{\sqrt{X}\,dx}{v^{m-2}}$$
$$=\frac{1}{(m-1)b'^2}\left(-\frac{b'\sqrt{X}}{v^{m-1}}+\tfrac{1}{2}\beta\int\frac{dx}{v^{m-1}\sqrt{X}}+c\int\frac{dx}{v^{m-2}\sqrt{X}}\right)$$
$$=\frac{1}{(m-2)b'^2}\left(-\frac{b'\sqrt{X}}{v^{m-1}}-k\int\frac{dx}{v^m\sqrt{X}}-\tfrac{1}{2}\beta\int\frac{dx}{v^{m-1}\sqrt{X}}\right).$$

208. $$\int v^m\sqrt{X}\,dx = \frac{1}{(m+2)c}\Big(b'v^{m-1}X\sqrt{X} - (m+\tfrac{1}{2})\beta\int v^{m-1}\sqrt{X}\,dx - (m-1)k\int v^{m-2}\sqrt{X}\,dx\Big).$$

209. $$\int\frac{dx}{v^m X^n\sqrt{X}} = -\frac{1}{(m-1)k}\Big(\frac{b'\sqrt{X}}{v^{m-1}X^n} + (m+n-\tfrac{3}{2})\beta\int\frac{dx}{v^{m-1}X^n\sqrt{X}} + (m+2n-2)c\int\frac{dx}{v^{m-2}X^n\sqrt{X}}\Big), \text{ if } k\neq 0.$$

210. $$\int\frac{dx}{v^m X^n\sqrt{X}} = \frac{-2}{(2m+2n-1)\beta}\Big(\frac{b'\sqrt{X}}{v^m X^n} + (m+2n-1)c\int\frac{dx}{v^{m-1}X^n\sqrt{X}}\Big); \text{ if } k=0.$$

211. $$\int\frac{X^n\,dx}{v^m\sqrt{X}}$$
$$= -\frac{1}{(m-1)k}\Big(\frac{b'X^n\sqrt{X}}{v^{m-1}} + (m-n-\tfrac{3}{2})\beta\int\frac{X^n\,dx}{v^{m-1}\sqrt{X}} + (m-2n-2)c\int\frac{X^n\,dx}{v^{m-2}\sqrt{X}}\Big)$$
$$= -\frac{1}{(m-2n)b'^2}\Big(\frac{b'X^{n-1}\sqrt{X}}{v^{m-1}} + (2n-1)k\int\frac{X^{n-1}dx}{v^m\sqrt{X}} + (n-\tfrac{1}{2})\beta\int\frac{X^{n-1}dx}{v^{m-1}\sqrt{X}}\Big)$$
$$= \frac{1}{(m-1)b'^2}\Big(-\frac{b'X^{n-1}\sqrt{X}}{v^{m-1}} + (n-\tfrac{1}{2})\beta\int\frac{X^{n-1}dx}{v^{m-1}\sqrt{X}} + (2n-1)c\int\frac{X^{n-1}dx}{v^{m-2}\sqrt{X}}\Big).$$

212. $$\int \frac{v^m X^n\,dx}{\sqrt{X}} = \frac{1}{(m+2n)c}\left(b'v^{m-1}X^n\sqrt{X} - (m+n-\tfrac{1}{2})\beta\int\frac{v^{m-1}X^n\,dx}{\sqrt{X}} - (m-1)k\int\frac{v^{m-2}X^n\,dx}{\sqrt{X}}\right).$$

213. $$\int \frac{v^m\,dx}{X^n\sqrt{X}} = \frac{1}{(m-2n)c}\left(\frac{b'v^{m-1}\sqrt{X}}{X^n} - (m-n-\tfrac{1}{2})\beta\int\frac{v^{m-1}\,dx}{X^n\sqrt{X}} - (m-1)k\int\frac{v^{m-2}\,dx}{X^n\sqrt{X}}\right).$$

IV. MISCELLANEOUS ALGEBRAIC EXPRESSIONS.

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214. $$\int \sqrt{2\,ax - x^2}\cdot dx = \frac{x-a}{2}\sqrt{2\,ax - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}.$$

215. $$\int \frac{dx}{\sqrt{2\,ax - x^2}} = \text{versin}^{-1}\frac{x}{a} = \cos^{-1}\left(1 - \frac{x}{a}\right)$$
$$= 2\sin^{-1}\sqrt{\frac{x}{2\,a}}.$$

216. $$\int \frac{x^n\,dx}{\sqrt{2\,ax - x^2}} = -\frac{x^{n-1}\sqrt{2\,ax - x^2}}{n}$$
$$- \frac{a\,(1 - 2\,n)}{n}\int \frac{x^{n-1}\,dx}{\sqrt{2\,ax - x^2}}.$$

217. $$\int \frac{dx}{x^n\sqrt{2\,ax - x^2}} = \frac{\sqrt{2\,ax - x^2}}{a\,(1 - 2\,n)\,x^n}$$
$$+ \frac{n-1}{(2\,n-1)\,a}\int \frac{dx}{x^{n-1}\sqrt{2\,ax - x^2}}.$$

218. $$\int x^n\sqrt{2\,ax - x^2}\cdot dx = -\frac{x^{n-1}\sqrt{(2\,ax - x^2)^3}}{n+2}$$
$$+ \frac{(2\,n+1)\,a}{n+2}\int x^{n-1}\sqrt{2\,ax - x^2}\cdot dx.$$

219. $$\int \frac{\sqrt{2\,ax - x^2}\cdot dx}{x^n} = \frac{\sqrt{(2\,ax - x^2)^3}}{(3 - 2\,n)\,ax^n}$$
$$+ \frac{n-3}{(2\,n-3)\,a}\int \frac{\sqrt{2\,ax - x^2}\cdot dx}{x^{n-1}}.$$

220. $$\int \frac{dx}{x\sqrt{x^n - a^2}} = \frac{2}{an}\sec^{-1}\left(\frac{x^{\frac{n}{2}}}{a}\right).$$

221. $$\int \frac{dx}{x\sqrt{x^n + a^2}} = \frac{1}{an} \log \frac{\sqrt{a^2 + x^n} - a}{\sqrt{a^2 + x^n} + a}.$$

222. $$\int \frac{x^{\frac{1}{2}}\,dx}{\sqrt{a^3 - x^3}} = \tfrac{2}{3} \sin^{-1}\left(\frac{x}{a}\right)^{\frac{3}{2}}.$$

223. $$\int \frac{dx}{(a + bx^2)\sqrt{x}} = \frac{1}{b\delta^3\sqrt{2}} \left\{ \log\left(\frac{x + \delta^2 + \sqrt{2\,\delta^2 x}}{\sqrt{a + bx^2}}\right) + \tan^{-1} \frac{\delta\sqrt{2\,x}}{\delta^2 - x} \right\}, \text{ where } b\delta^4 = a.$$

224. $$\int \frac{\sqrt{x}\cdot dx}{a + bx^2} = \frac{1}{b\delta\sqrt{2}} \left\{ \tan^{-1} \frac{\sqrt{2\,\delta^2 x}}{\delta^2 - x} - \log\left(\frac{x + \delta^2 + \sqrt{2\,\delta^2 x}}{\sqrt{a + bx^2}}\right) \right\}, \text{ where } b\delta^4 = a.$$

225. $$\int \frac{x^{\frac{3}{2}}\cdot dx}{a + bx^2} = \frac{2\sqrt{x}}{b} - \frac{a}{b}\int \frac{dx}{(a + bx^2)\sqrt{x}}.$$

226. $$\int \frac{dx}{(a + bx^2)^2\sqrt{x}} = \frac{\sqrt{x}}{2\,a(a + bx^2)} + \frac{3}{4\,a}\int \frac{dx}{(a + bx^2)\sqrt{x}}.$$

227. $$\int \frac{\sqrt{x}\cdot dx}{(a + bx^2)^2} = \frac{x^{\frac{3}{2}}}{2\,a(a + bx^2)} + \frac{1}{4\,a}\int \frac{\sqrt{x}\cdot dx}{(a + bx^2)}.$$

If a_1, a_2, a_3, etc., are the roots of the equation

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0,$$

the integrand in the expression

$$\int \frac{(q_0x^m + q_1x^{m-1} + \cdots + q_n)\,dx}{(p_0x^n + p_1x^{n-1} + \cdots + p_n)\sqrt{a + bx + cx^2}},$$

where $m < n$, may be expressed as the sum of a number of partial fractions of the form $\dfrac{A}{(x - a_k)^r \sqrt{a + bx + cx^2}}$, and these can be integrated by the aid of equations given above. Thus,

228. $$\int \frac{(px + q)\,dx}{(x - a')(x - b')\sqrt{a + bx + cx^2}}$$
$$= \frac{q + a'p}{a' - b'} \int \frac{dx}{(x - a')\sqrt{a + bx + cx^2}}$$
$$- \frac{q + b'p}{a' - b'} \int \frac{dx}{(x - b')\sqrt{a + bx + cx^2}}.$$

229. $$\int \frac{dx}{(a' + c'x^2)\sqrt{a + cx^2}}$$
$$= \frac{1}{\sqrt{a'(ac' - a'c)}} \tan^{-1} x \sqrt{\frac{(ac' - a'c)}{a'(a + cx^2)}}$$
$$= \frac{1}{2\sqrt{a'(a'c - ac')}} \log \frac{\sqrt{a'(a + cx^2)} + \sqrt{a'c - ac'}}{\sqrt{a'(a + cx^2)} - \sqrt{a'c - ac'}}.$$

230. $$\int \frac{x\,dx}{(a' + c'x^2)\sqrt{a + cx^2}}$$
$$= \frac{1}{\sqrt{c'(a'c - ac')}} \tan^{-1} \sqrt{\frac{c'(a + cx^2)}{a'c - ac'}}$$
$$= \frac{1}{2\sqrt{c'(ac' - a'c)}} \log \frac{\sqrt{c'(a + cx^2)} - \sqrt{ac' - a'c}}{\sqrt{c'(a + cx^2)} + \sqrt{ac' - a'c}}.$$

231. $$\int f\left\{x,\ \sqrt[n]{\frac{a + bx}{a' + b'x}}\right\} dx$$
$$= n(a'b - ab') \int f\left(\frac{a - a'z^n}{b'z^n - b},\ z\right) \cdot \frac{z^{n-1}\,dz}{(b'z^n - b)^2},$$

where $z^n(a' + b'x) = a + bx$.

232. $\int f(x, \sqrt[n]{c + \sqrt[m]{a + bx}})\, dx$

$$= \frac{mn}{b} \int f\left\{ \frac{(z^n - c)^m - a}{b},\ z \right\} (z^n - c)^{m-1} z^{n-1}\, dz,$$

where $z^n = c + \sqrt[m]{a + bx}$.

233. $\int f\left\{ x,\ \left[\frac{a + bx}{a' + b'x} \right]^{\frac{m}{n}},\ \left[\frac{a + bx}{a' + b'x} \right]^{\frac{p}{q}},\ \cdots \right\} dx$

$$= s(a'b - ab') \int f\left\{ \frac{a'y^s - a}{b - b'y^s},\ y^{\frac{ms}{n}},\ y^{\frac{ps}{q}},\ \cdots \right\} \frac{y^{s-1}dy}{(b - b'y^s)^2},$$

where $y^s(a' + b'x) = a + bx$ and s is the least common multiple of n, q, etc.

234. $\int f(x, \sqrt{a + bx + x^2})\, dx$

$$= 2 \int f\left(\frac{2\sqrt{a} \cdot z - b}{1 - z^2},\ \frac{z^2\sqrt{a} - bz + \sqrt{a}}{1 - z^2} \right) \cdot \frac{(z^2\sqrt{a} - bz + \sqrt{a})\, dz}{(1 - z^2)^2},$$

where $xz + \sqrt{a} = \sqrt{a + bx + x^2}$.

235. $\int f(x, \sqrt{a + bx + x^2})\, dx$

$$= \int f\left(\frac{u^2 - a}{b - 2u},\ \frac{u^2 - bu + a}{2u - b} \right) \frac{2(bu - a - u^2)\, du}{(b - 2u)^2},$$

where $u = \sqrt{a + bx + x^2} - x$.

V. TRANSCENDENTAL FUNCTIONS.

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236. $\int \sin x \cdot f(\cos x)\, dx = -\int f(\cos x)\, d \cos x.$

237. $\int \cos x \cdot f(\sin x)\, dx = \int f(\sin x)\, d \sin x.$

238. $\int \sin x \cdot f(\sin x,\ \cos x)\, dx = -\int f(\sqrt{1-z^2},\ z)\, dz,$

where $z = \cos x$.

239. $\int \cos x \cdot f(\sin x,\ \cos x)\, dx = \int f(z,\ \sqrt{1-z^2})\, dz,$

where $z = \sin x$.

240. $\int f(\sin x,\ \cos x)\, dx = \int f(z,\ \sqrt{1-z^2}) \frac{dz}{\sqrt{1-z^2}};\ z = \sin x.$

241. $\int f(\sin x)\, dx = -\int f\left(\cos\left(\frac{\pi}{2} - x\right)\right) d\left(\frac{\pi}{2} - x\right).$

242. $\int f(\tan x)\, dx = -\int f \operatorname{ctn}\left(\frac{\pi}{2} - x\right) d\left(\frac{\pi}{2} - x\right).$

243. $\int f(\sec x)\, dx = -\int f \csc\left(\frac{\pi}{2} - x\right) d\left(\frac{\pi}{2} - x\right).$

244. $\int \frac{\sin x \cdot f(\sin^2 x)\, dx}{\sqrt{1 - k^2 \sin^2 x}} = \int \frac{f(z)\, dz}{2\sqrt{(1-z)(1-k^2 z)}},$

where $z = \sin^2 x$.

245. $\int \frac{\cos x \cdot f(\cos^2 x)\, dx}{\sqrt{1 - k^2 \sin^2 x}} = \int \frac{f(1-z)\, dz}{2\sqrt{z(1-k^2 z)}},$ where $z = \sin^2 x$.

246. $\displaystyle\int\frac{\tan x\cdot f(\tan^2 x)\,dx}{\sqrt{1-k^2\sin^2 x}}=\int f\left(\frac{z}{1-z}\right)\frac{dz}{2\,(1-z)\sqrt{1-k^2 z}},$

where $z=\sin^2 x$.

247. $\displaystyle\int f(\tan x)\,dx=\int\frac{f(z)\,dz}{1+z^2},$ where $z=\tan x$.

248. $\displaystyle\int\sec^{n+2}x\cdot f(\tan x)\,dx=\int(1+z^2)^{\frac{n}{2}}f(z)\,dz;$ $z=\tan x$.

249. $\displaystyle\int f(\sin x,\ \cos x)\,dx$

$$=-\int f\left(\cos\left(\frac{\pi}{2}-x\right),\ \sin\left(\frac{\pi}{2}-x\right)\right)d\left(\frac{\pi}{2}-x\right)\cdot$$

250. $\displaystyle\int f(x)\cdot\sin^{-1}x\cdot dx=\sin^{-1}x\cdot\phi(x)-\int\frac{\phi(x)\,dx}{\sqrt{1-x^2}},\ dx,$

where $\displaystyle\phi(x)=\int f(x)\,dx$.

251. $\displaystyle\int f(x)\cdot\cos^{-1}x\,dx=\cos^{-1}x\cdot\phi(x)+\int\frac{\phi(x)\,dx}{\sqrt{1-x^2}}\cdot$

252. $\displaystyle\int f(x)\cdot\tan^{-1}x\,dx=\tan^{-1}x\cdot\phi(x)-\int\frac{\phi(x)\,dx}{1+x^2}\cdot$

253. $\displaystyle\int f(x)\cdot\operatorname{ctn}^{-1}x\,dx=\operatorname{ctn}^{-1}x\cdot\phi(x)+\int\frac{\phi(x)\,dx}{1+x^2}\cdot$

254. $\displaystyle\int f(x,\ \cos x)\,dx=-\int f\left(\frac{\pi}{2}-z,\ \sin z\right)dz,$

where $z=\dfrac{\pi}{2}-x$.

255. $\displaystyle\int\frac{\sin x\cdot f(\cos x)\,dx}{a+b\cos x}=-\frac{1}{b}\int f\left(\frac{z-a}{b}\right)\frac{dz}{z},$

where $z=a+b\cos x$.

256. $\int f(x,\ \log x)\,dx = \int f(e^z,\ z)\,e^z\,dz,$ where $z = \log x.$

257. $\int \frac{f(\log x)\,dx}{x} = \int f(z)\,dz,$ where $z = \log x.$

258. $\int x^m f(\log x)\,dx = \int e^{(m+1)z} f(z)\,dz.$

259. $\int f(\sin x,\ \cos x,\ \tan x,\ \operatorname{ctn} x,\ \sec x,\ \csc x)\,dx$

$$= 2\int f\left(\frac{2z}{1+z^2}, \frac{1-z^2}{1+z^2}, \frac{2z}{1-z^2}, \frac{1-z^2}{2z}, \frac{1+z^2}{1-z^2}, \frac{1+z^2}{2z}\right)$$

$\frac{dz}{1+z^2},$ where $z = \tan\frac{x}{2};$

$$= \int f\left(z,\ \sqrt{1-z^2}, \frac{z}{\sqrt{1-z^2}}, \frac{\sqrt{1-z^2}}{z}, \frac{1}{\sqrt{1-z^2}}, \frac{1}{z}\right)$$

$\frac{dz}{\sqrt{1-z^2}},$ where $z = \sin x;$

$$= \int f\left(\frac{z}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+z^2}},\ z,\ \frac{1}{z},\ \sqrt{1+z^2}, \frac{\sqrt{1+z^2}}{z}\right)$$

$\frac{dz}{1+z^2},$ where $z = \tan x;$

$$= \int f\left(\sqrt{z},\ \sqrt{1-z},\ \sqrt{\frac{z}{1-z}},\ \sqrt{\frac{1-z}{z}},\ \frac{1}{\sqrt{1-z}},\ \frac{1}{\sqrt{z}}\right)$$

$\frac{dz}{2\sqrt{z}(1-z)},$ where $z = \sin^2 x;$

$$= \int f\left(\sqrt{\frac{z}{1+z}},\ \frac{1}{\sqrt{1+z}},\ \sqrt{z},\ \frac{1}{\sqrt{z}},\ \sqrt{1+z},\ \sqrt{\frac{1+z}{z}}\right)$$

$\frac{dz}{2\sqrt{z}(1+z)},$ where $z = \tan^2 x.$

260. $\int \sin x\, dx = -\cos x.$

261. $\int \sin^2 x\, dx = -\tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}x = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x.$

262. $\int \sin^3 x\, dx = -\tfrac{1}{3}\cos x(\sin^2 x + 2).$

263. $\int \sin^n x\, dx = -\frac{\sin^{n-1}x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx.$

264. $\int \cos x\, dx = \sin x.$

265. $\int \cos^2 x\, dx = \tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x.$

266. $\int \cos^3 x\, dx = \tfrac{1}{3}\sin x(\cos^2 x + 2).$

267. $\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$

268. $\int \sin x \cos x\, dx = \tfrac{1}{2}\sin^2 x.$

269. $\int \sin^2 x \cos^2 x\, dx = -\tfrac{1}{8}(\tfrac{1}{4}\sin 4x - x).$

270. $\int \sin x \cos^m x\, dx = -\frac{\cos^{m+1}x}{m+1}.$

271. $\int \sin^m x \cos x\, dx = \frac{\sin^{m+1}x}{m+1}.$

272. $$\int \cos^m x \sin^n x\, dx = \frac{\cos^{m-1}x \sin^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \cos^{m-2}x \sin^n x\, dx.$$

273. $$\int \cos^m x \sin^n x\, dx = -\frac{\sin^{n-1}x \cos^{m+1}x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2}x\, dx.$$

274. $$\int\frac{\sin^n x\,dx}{\cos^m x}=\frac{1}{n-m}\left(-\frac{\sin^{n-1}x}{\cos^{m-1}x}+(n-1)\int\frac{\sin^{n-2}x\,dx}{\cos^m x}\right)$$
$$=\frac{1}{m-1}\left(\frac{\sin^{n+1}x}{\cos^{m-1}x}-(n-m+2)\int\frac{\sin^n x\,dx}{\cos^{m-2}x}\right)$$
$$=\frac{1}{m-1}\left(\frac{\sin^{n-1}x}{\cos^{m-1}x}-(n-1)\int\frac{\sin^{n-2}x\,dx}{\cos^{m-2}x}\right).$$

275. $$\int\frac{\cos^m x\,dx}{\sin^n x}=-\frac{\cos^{m+1}x}{(n-1)\sin^{n-1}x}-\frac{m-n+2}{n-1}\int\frac{\cos^m x\,dx}{\sin^{n-2}x}$$
$$=\frac{\cos^{m-1}x}{(m-n)\sin^{n-1}x}+\frac{m-1}{m-n}\int\frac{\cos^{m-2}x\,dx}{\sin^n x}$$
$$=-\frac{1}{n-1}\,\frac{\cos^{m-1}x}{\sin^{n-1}x}-\frac{m-1}{n-1}\int\frac{\cos^{m-2}x\,dx}{\sin^{n-2}x}.$$

276. $$\int\frac{\sin^m x\,dx}{\cos^n x}=-\int\frac{\cos^m\left(\frac{\pi}{2}-x\right)d\left(\frac{\pi}{2}-x\right)}{\sin^n\left(\frac{\pi}{2}-x\right)}.$$

277. $$\int\frac{dx}{\sin x\cos x}=\log\tan x.$$

278. $$\int\frac{dx}{\cos x\sin^2 x}=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)-\csc x.$$

279. $$\int\frac{dx}{\sin^m x\cos^n x}$$
$$=\frac{1}{n-1}\cdot\frac{1}{\sin^{m-1}x\cdot\cos^{n-1}x}+\frac{m+n-2}{n-1}\int\frac{dx}{\sin^m x\cdot\cos^{n-2}x}$$
$$=-\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1}x\cdot\cos^{n-1}x}+\frac{m+n-2}{m-1}\int\frac{dx}{\sin^{m-2}x\cdot\cos^n x}.$$

280. $$\int\frac{dx}{\sin^m x}=-\frac{1}{m-1}\cdot\frac{\cos x}{\sin^{m-1}x}+\frac{m-2}{m-1}\int\frac{dx}{\sin^{m-2}x}.$$

281. $$\int \frac{dx}{\cos^n x} = \frac{1}{n-1}\cdot\frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int\frac{dx}{\cos^{n-2} x}.$$

282. $$\int \tan x\, dx = -\log\cos x.$$

283. $$\int \tan^2 x\, dx = \tan x - x.$$

284. $$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx.$$

285. $$\int \operatorname{ctn} x\, dx = \log\sin x.$$

286. $$\int \operatorname{ctn}^2 x\, dx = -\operatorname{ctn} x - x.$$

287. $$\int \operatorname{ctn}^n x\, dx = -\frac{\operatorname{ctn}^{n-1} x}{n-1} - \int \operatorname{ctn}^{n-2} x\, dx.$$

288. $$\int \sec x\, dx = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \tfrac{1}{2}\log\frac{1+\sin x}{1-\sin x}.$$

289. $$\int \sec^2 x\, dx = \tan x.$$

290. $$\int \sec^n x\, dx = \int\frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\int\frac{dx}{\cos^{n-2} x}$$
$$= \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx.$$

291. $$\int \csc x\, dx = \log\tan\tfrac{1}{2}x.$$

292. $$\int \csc^2 x\, dx = -\operatorname{ctn} x.$$

293. $$\int \csc^n x\,dx = \int \frac{dx}{\sin^n x}$$

$$= -\frac{\cos x}{(n-1)\sin^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2}x}$$

$$= -\frac{\cos x}{(n-1)\sin^{n-1}x} + \frac{n-2}{n-1}\int \csc^{n-2}x\,dx.$$

$x = \frac{\pi}{2} - y$

$y = \frac{\pi}{2} - x$

294. $$\int \frac{dx}{1+\sin x} = -\tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}x\right).$$

295. $$\int \frac{dx}{1-\sin x} = \operatorname{ctn}\left(\tfrac{1}{4}\pi - \tfrac{1}{2}x\right) = \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}x\right).$$

$\int \frac{dx}{1+\sin x}$ $\int \frac{-dy}{1+\cos y}$

296. $$\int \frac{dx}{1+\cos x} = \tan \tfrac{1}{2}x, \text{ or } \csc x - \operatorname{ctn} x.$$

297. $$\int \frac{dx}{1-\cos x} = -\operatorname{ctn}\tfrac{1}{2}x, \text{ or } -\operatorname{ctn} x - \csc x.$$

$= -\tan\theta$

298. $$\int \frac{d\,x}{a \pm b\sin x} = \frac{2\sec\theta}{a}\cdot\tan^{-1}\left(\sec\theta\cdot\tan\tfrac{1}{2}x \pm \tan\theta\right),$$

$= -\tan\left(\frac{\pi}{4}\right.$

if $a > b$, and $b = a\sin\theta$.

299. $$\int \frac{dx}{a \pm b\sin x} = \frac{\pm\sec\alpha}{b}\log\frac{\sin\frac{1}{2}(\alpha \pm x)}{\cos\frac{1}{2}(x \mp \alpha)},$$

if $b > a$, and $a = b\sin\alpha$.

300. $$\int \frac{dx}{a + b\cos x} = \frac{-1}{\sqrt{a^2-b^2}}\cdot\sin^{-1}\left[\frac{b + a\cos x}{a + b\cos x}\right],$$

$$\text{or } \frac{1}{\sqrt{a^2-b^2}}\sin^{-1}\left[\frac{\sqrt{a^2-b^2}\cdot\sin x}{a + b\cos x}\right],$$

$$\text{or } \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\tfrac{1}{2}x\right],$$

$$\text{or } \frac{1}{\sqrt{a^2-b^2}}\tan^{-1}\left[\frac{\sqrt{a^2-b^2}\cdot\sin x}{b + a\cos x}\right],$$

$$\text{or } \frac{1}{\sqrt{b^2-a^2}}\log\left[\frac{b+a\cos x+\sqrt{b^2-a^2}\cdot\sin x}{a+b\cos x}\right],$$

$$\text{or } \frac{1}{\sqrt{b^2-a^2}}\log\left[\frac{\sqrt{b+a}+\sqrt{b-a}\cdot\tan\tfrac{1}{2}x}{\sqrt{b+a}-\sqrt{b-a}\cdot\tan\tfrac{1}{2}x}\right],$$

$$\text{or } \frac{1}{\sqrt{b^2-a^2}}\tanh^{-1}\left[\frac{\sqrt{b^2-a^2}\cdot\sin x}{b+a\cos x}\right].$$

301. $$\int\frac{dx}{a+b\tan x}=\frac{1}{a^2+b^2}[b\log(a\cos x+b\sin x)+ax].$$

302. $$\int\frac{dx}{\sin x+\cos x}=\frac{1}{\sqrt{2}}\log\tan\left(\tfrac{1}{2}x+\tfrac{1}{8}\pi\right).$$

303. $$\int\frac{\sin x\,dx}{a+b\cos x}=-\int\frac{\cos\left(\frac{1}{2}\pi-x\right)d\left(\frac{1}{2}\pi-x\right)}{a+b\sin\left(\frac{1}{2}\pi-x\right)}$$

$$=-\frac{1}{b}\log(a+b\cos x).$$

304. $$\int\frac{(a'+b'\cos x)\,dx}{a+b\cos x}=\frac{b'x}{b}+\frac{a'b-ab'}{b}\int\frac{dx}{a+b\cos x}.$$

305. $$\int\frac{(a'+b'\cos x)\,dx}{(a+b\cos x)^2}=\frac{ab'-a'b}{a^2-b^2}\,\frac{\sin x}{a+b\cos x}$$

$$+\frac{aa'-bb'}{a^2-b^2}\int\frac{dx}{a+b\cos x}.$$

306. $$\int\frac{(a'+b'\cos x)\,dx}{(a+b\cos x)^n}=\frac{1}{(n-1)(a^2-b^2)}\left[\frac{(ab'-a'b)\sin x}{(a+b\cos x)^{n-1}}\right.$$

$$\left.+\int\frac{[(aa'-bb')(n-1)+(n-2)(ab'-a'b)\cos x]\,dx}{(a+b\cos x)^{n-1}}\right].$$

307. $$\int \frac{(a' + b' \cos x)\, dx}{(1 + \cos x)^n} = \frac{(a' - b') \tan \frac{1}{2} x}{(2n - 1)(1 + \cos x)^{n-1}} + \frac{n(a' + b') - a'}{2n - 1} \int \frac{dx}{(1 + \cos x)^{n-1}}.$$

308. $$\int \frac{dx}{(a + b \cos x)^n} = \frac{1}{(n - 1)(a^2 - b^2)} \left[\frac{-b \sin x}{(a + b \cos x)^{n-1}} + (2n - 3)\, a \int \frac{dx}{(a + b \cos x)^{n-1}} - (n - 2) \int \frac{dx}{(a + b \cos x)^{n-2}} \right].$$

309. $$\int \frac{dx}{(1 + \cos x)^n} = \frac{\tan \frac{1}{2} x}{(2n - 1)(1 + \cos x)^{n-1}} + \frac{n - 1}{2n - 1} \int \frac{dx}{(1 + \cos x)^{n-1}}.$$

310. $$\int \frac{(a' + b' \cos x)\, dx}{\sin x\, (a + b \cos x)} = \frac{a'b - ab'}{a^2 - b^2} \log (a + b \cos x) + \frac{a' + b'}{a + b} \log \sin \tfrac{1}{2} x - \frac{a' - b'}{a - b} \log \cos \tfrac{1}{2} x.$$

311. $$\int \frac{(a' + b' \cos x)\, dx}{\cos x\, (a + b \cos x)} = \frac{a'}{a} \log \tan \tfrac{1}{2} (\tfrac{1}{2} \pi + x) + \frac{(ab' - a'b)}{a} \int \frac{dx}{a + b \cos x}.$$

312. $$\int \frac{(a' + b' \cos x)\, dx}{\sin x\, (1 \pm \cos x)} = \pm \frac{\frac{1}{2} (a' \mp b')}{1 \pm \cos x} + \tfrac{1}{2} (a' \pm b') \log \tan \tfrac{1}{2} x.$$

313. $$\int \frac{dx}{(1 - \cos x)^n} = \frac{-\operatorname{ctn} \frac{1}{2} x}{(2n - 1)(1 - \cos x)^{n-1}} + \frac{n - 1}{2n - 1} \int \frac{dx}{(1 - \cos x)^{n-1}}.$$

314. $$\int \frac{dx}{a^2 - b^2 \cos^2 x} = \int \frac{dx}{(a^2 - b^2) + b^2 \sin^2 x}$$

$$= \frac{1}{2\,ab \sin a} \log \frac{\sin(a - x)}{\sin(a + x)},$$

or $\dfrac{1}{a^2 \sin\beta} \tan^{-1}\left(\dfrac{\tan x}{\sin\beta}\right)$, where $\cos a = \dfrac{1}{\cos\beta} = \dfrac{a}{b}$.

315. $$\int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{ab} \tan^{-1}\left(\frac{b \tan x}{a}\right).$$

316. $$\int \frac{\sin^2 x\, dx}{a + b \cos^2 x} = \frac{\sqrt{a + b}}{b\sqrt{a}} \tan^{-1}\left(\tan x \cdot \sqrt{\frac{a}{a + b}}\right) - \frac{x}{b}.$$

317. $$\int \frac{\sin x \cos x\, dx}{a \cos^2 x + b \sin^2 x} = \frac{1}{2(b - a)} \log(a \cos^2 x + b \sin^2 x).$$

318. $$\int \frac{dx}{(a + b \cos x + c \sin x)^n} = \int \frac{d(x - a)}{[a + r \cos(x - a)]^n},$$

where $b = r \cos a$ and $c = r \sin a$.

319. $$\int \frac{dx}{a + b \cos x + c \sin x}$$

$$= \frac{-1}{\sqrt{a^2 - b^2 - c^2}} \cdot \sin^{-1}\left[\frac{b^2 + c^2 + a(b \cos x + c \sin x)}{\sqrt{(b^2 + c^2)}(a + b \cos x + c \sin x)}\right]$$

$$= \frac{1}{\sqrt{b^2 + c^2 - a^2}} \cdot \log$$

$$\left[\frac{b^2 + c^2 + a(b \cos x + c \sin x) + \sqrt{b^2 + c^2 - a^2}\,(b \sin x - c \cos x)}{\sqrt{(b^2 + c^2)}\,(a + b \cos x + c \sin x)}\right]$$

$$= \frac{1}{\sqrt{b^2 + c^2 - a^2}} \cdot \log \frac{\sqrt{b^2 + c^2 - a^2} - c + (b - a) \tan \frac{1}{2} x}{\sqrt{b^2 + c^2 - a^2} + c - (b - a) \tan \frac{1}{2} x}$$

$$= \frac{2}{\sqrt{a^2 - b^2 - c^2}} \tan^{-1}\left[\frac{(a - b) \tan \frac{1}{2} x + c}{\sqrt{a^2 - b^2 - c^2}}\right].$$

320. $$\int \frac{dx}{a(1+\cos x)+c\sin x}=\frac{1}{c}\log\left(a+c\tan\tfrac{1}{2}x\right).$$

321. $$\int \frac{dx}{(a[1+\cos x]+c\sin x)^2}$$
$$=\frac{1}{c^3}\left[\frac{c(a\sin x-c\cos x)}{a(1+\cos x)+c\sin x}-a\log\left(a+c\tan\tfrac{1}{2}x\right)\right].$$

322. $$\int \frac{(x+\sin x)\,dx}{1+\cos x}=x\tan\tfrac{1}{2}x.$$

323. $$\int \cos x\sqrt{1-k^2\sin^2x}\,dx$$
$$=\tfrac{1}{2}\sin x\sqrt{1-k^2\sin^2x}+\frac{1}{2k}\sin^{-1}(k\sin x).$$

324. $$\int \sin x\sqrt{1-k^2\sin^2x}\,dx$$
$$=-\tfrac{1}{2}\cos x\sqrt{1-k^2\sin^2x}-\frac{1-k^2}{2k}\log\left(k\cos x+\sqrt{1-k^2\sin^2x}\right).$$

325. $$\int \sin x(1-k^2\sin^2x)^{\frac{3}{2}}\,dx=-\tfrac{1}{4}\cos x(1-k^2\sin^2x)^{\frac{3}{2}}$$
$$+\tfrac{3}{4}(1-k^2)\int \sin x\sqrt{1-k^2\sin^2x}\,dx.$$

326. $$\int \frac{\cos x\,dx}{\sqrt{1-k^2\sin^2x}}=\frac{1}{k}\sin^{-1}(k\sin x),$$
$$\text{or }\frac{1}{b}\log\left(b\sin x+\sqrt{1+b^2\sin^2x}\right),\ \text{where } b^2=-k^2.$$

327. $$\int \frac{\sin x\,dx}{\sqrt{1-k^2\sin^2x}}=-\frac{1}{k}\log\left(k\cos x+\sqrt{1-k^2\sin^2x}\right),$$
$$\text{or }-\frac{1}{b}\sin^{-1}\frac{b\cos x}{\sqrt{1+b^2}},\ \text{where } b^2=-k^2$$

328. $$\int \frac{\tan x\,dx}{\sqrt{1-k^2\sin^2x}}$$
$$=\frac{1}{2\sqrt{1-k^2}}\log\left(\frac{\sqrt{1-k^2\sin^2x}+\sqrt{1-k^2}}{\sqrt{1-k^2\sin^2x}-\sqrt{1-k^2}}\right).$$

329. $$\int\frac{x\,dx}{1+\sin x}=-x\tan\tfrac{1}{2}(\tfrac{1}{2}\pi-x)+2\log\cos\tfrac{1}{2}(\tfrac{1}{2}\pi-x).$$

330. $$\int\frac{x\,dx}{1-\sin x}=x\operatorname{ctn}\tfrac{1}{2}(\tfrac{1}{2}\pi-x)+2\log\sin\tfrac{1}{2}(\tfrac{1}{2}\pi-x).$$

331. $$\int\frac{x\,dx}{1+\cos x}=x\tan\tfrac{1}{2}x+2\log\cos\tfrac{1}{2}x.$$

332. $$\int\frac{x\,dx}{1-\cos x}=-x\operatorname{ctn}\tfrac{1}{2}x+2\log\sin\tfrac{1}{2}x.$$

333. $$\int\frac{\tan x\,dx}{\sqrt{a+b\tan^2 x}}=\frac{1}{\sqrt{b-a}}\cos^{-1}\left(\frac{\sqrt{b-a}}{\sqrt{b}}\cdot\cos x\right).$$

334. $$\int\frac{dx}{a+b\tan^2 x}=\frac{1}{a-b}\left[x-\sqrt{\frac{b}{a}}\cdot\tan^{-1}\left(\sqrt{\frac{b}{a}}\cdot\tan x\right)\right].$$

335. $$\int\frac{\tan x\,dx}{a+b\tan x}$$
$$=\frac{1}{a^2+b^2}\left\{bx-a\log(a+b\tan x)+a\log\sec x\right\}.$$

336. $$\int x\sin x\,dx=\sin x-x\cos x.$$

337. $$\int x^2\sin x\,dx=2x\sin x-(x^2-2)\cos x.$$

338. $$\int x^3\sin x\,dx=(3x^2-6)\sin x-(x^3-6x)\cos x.$$

339. $$\int x^m\sin x\,dx=-x^m\cos x+m\int x^{m-1}\cos x\,dx.$$

340. $$\int x\cos x\,dx=\cos x+x\sin x.$$

341. $$\int x^2\cos x\,dx=2x\cos x+(x^2-2)\sin x.$$

342. $$\int x^3\cos x\,dx=(3x^2-6)\cos x+(x^3-6x)\sin x.$$

343. $$\int x^m \cos x\,dx = x^m \sin x - m\int x^{m-1}\sin x\,dx.$$

344. $$\int \frac{\sin x}{x^m}dx = -\frac{1}{m-1}\cdot\frac{\sin x}{x^{m-1}} + \frac{1}{m-1}\int\frac{\cos x}{x^{m-1}}dx.$$

345. $$\int \frac{\cos x}{x^m}dx = -\frac{1}{m-1}\cdot\frac{\cos x}{x^{m-1}} - \frac{1}{m-1}\int\frac{\sin x}{x^{m-1}}dx.$$

346. $$\int \frac{\sin x}{x}dx = x - \frac{x^3}{3\cdot 3\,!} + \frac{x^5}{5\cdot 5\,!} - \frac{x^7}{7\cdot 7\,!} + \frac{x^9}{9\cdot 9\,!}\cdots.$$

347. $$\int \frac{\cos x}{x}dx = \log x - \frac{x^2}{2\cdot 2\,!} + \frac{x^4}{4\cdot 4\,!} - \frac{x^6}{6\cdot 6\,!} + \frac{x^8}{8\cdot 8\,!}\cdots.$$

348. $$\int \frac{x\,dx}{\sin x} = x + \frac{x^3}{3\cdot 3\,!} + \frac{7\,x^5}{3\cdot 5\cdot 5\,!} + \frac{29\,x^7}{3\cdot 7\cdot 7\,!} + \frac{127\,x^9}{3\cdot 9\cdot 9\,!} + \cdots$$

349. $$\int \frac{x\,dx}{\cos x} = \frac{x^2}{2} + \frac{x^4}{4\cdot 2\,!} + \frac{5\,x^6}{6\cdot 4\,!} + \frac{61\,x^8}{8\cdot 6\,!} + \frac{1385\,x^{10}}{10\cdot 8\,!} + \cdots.$$

350. $$\int \frac{x\,dx}{\sin^2 x} = -x \operatorname{ctn} x + \log \sin x.$$

351. $$\int \frac{x\,dx}{\cos^2 x} = x\tan x + \log\cos x.$$

352. $$n^2\int x^m \sin^n x\,dx$$
$$= x^{m-1}\sin^{n-1}x\,(m\sin x - nx\cos x)$$
$$+ n(n-1)\int x^m \sin^{n-2}x\,dx - m(m-1)\int x^{m-2}\sin^n x\,dx.$$

353. $$n^2\int x^m \cos^n x\,dx$$
$$= x^{m-1}\cos^{n-1}x\,(m\cos x + nx\sin x)$$
$$+ n(n-1)\int x^m \cos^{n-2}x\,dx - m(m-1)\int x^{m-2}\cos^n x\,dx.$$

354. $\int \frac{x^m\,dx}{\sin^n x}$

$$= \frac{1}{(n-1)(n-2)}\left[-\frac{x^{m-1}(m\sin x + (n-2)x\cos x)}{\sin^{n-1}x}\right.$$

$$\left. + (n-2)^2\int\frac{x^m\,dx}{\sin^{n-2}x} + m(m-1)\int\frac{x^{m-2}\,dx}{\sin^{n-2}x}\right].$$

355. $\int \frac{x^m\,dx}{\cos^n x}$

$$= \frac{1}{(n-1)(n-2)}\left[-\frac{x^{m-1}(m\cos x - (n-2)x\sin x)}{\cos^{n-1}x}\right.$$

$$\left. + (n-2)^2\int\frac{x^m\,dx}{\cos^{n-2}x} + m(m-1)\int\frac{x^{m-2}\,dx}{\cos^{n-2}x}\right].$$

356. $\int \frac{\sin^n x\,dx}{x^m}$

$$= \frac{1}{(m-1)(m-2)}\left[-\frac{\sin^{n-1}x((m-2)\sin x + nx\cos x)}{x^{m-1}}\right.$$

$$\left. - n^2\int\frac{\sin^n x\,dx}{x^{m-2}} + n(n-1)\int\frac{\sin^{n-2}x\,dx}{x^{m-2}}\right].$$

357. $\int \frac{\cos^n x\,dx}{x^m}$

$$= \frac{1}{(m-1)(m-2)}\left[\frac{\cos^{n-1}x(nx\cos x - (m-2)\cos x)}{x^{m-1}}\right.$$

$$\left. - n^2\int\frac{\cos^n x\,dx}{x^{m-2}} + n(n-1)\int\frac{\cos^{n-2}x\,dx}{x^{m-2}}\right].$$

358. $\int x^p \sin^m x \cos^n x\,dx$

$$= \frac{1}{(m+n)^2}\left[x^{p-1}\sin^m x\cos^{n-1}x\,(p\cos x + (m+n)x\sin x)\right.$$

$$+ (n-1)(m+n)\int x^p \sin^m x\cos^{n-2}x\,dx$$

$$-mp\int x^{p-1}\sin^{m-1}x\cos^{n-1}x\,dx$$

$$\left.-p(p-1)\int x^{p-2}\sin^m x\cos^n x\,dx\right]\cdot$$

$$=\frac{1}{(m+n)^2}\Big[x^{p-1}\sin^{m-1}x\cos^n x(p\sin x-(m+n)x\cos x)$$

$$+(m-1)(m+n)\int x^p\sin^{m-2}x\cos^n x\,dx$$

$$+np\int x^{p-1}\sin^{m-1}x\cos^{n-1}x\,dx$$

$$\left.-p(p-1)\int x^{p-2}\sin^m x\cos^n x\,dx\right]\cdot$$

359. $$\int\sin mx\sin nx\,dx=\frac{\sin(m-n)x}{2(m-n)}-\frac{\sin(m+n)x}{2(m+n)}.$$

360. $$\int\sin mx\cos nx\,dx=-\frac{\cos(m-n)x}{2(m-n)}-\frac{\cos(m+n)x}{2(m+n)}.$$

361. $$\int\cos mx\cos nx\,dx=\frac{\sin(m-n)x}{2(m-n)}+\frac{\sin(m+n)x}{2(m+n)}.$$

362. $$\int\sin^2 mx\,dx=\frac{1}{2m}(mx-\sin mx\cos mx).$$

363. $$\int\cos^2 mx\,dx=\frac{1}{2m}(mx+\sin mx\cos mx).$$

364. $$\int\sin mx\cos mx\,dx=-\frac{1}{4m}\cos 2mx.$$

365. $$\int\sin nx\sin^m x\,dx=\frac{1}{m+n}\Big[-\cos nx\sin^m x$$

$$\left.+m\int\cos(n-1)x\cdot\sin^{m-1}x\,dx\right]\cdot$$

366. $$\int \sin nx \cos^m x\,dx = \frac{1}{m+n}\Big[-\cos nx \cos^m x + m\int \sin(n-1)x \cdot \cos^{m-1}x\,dx\Big].$$

367. $$\int \cos nx \sin^m x\,dx = \frac{1}{m+n}\Big[\sin nx \sin^m x - m\int \sin(n-1)x \cdot \sin^{m-1}x\,dx\Big].$$

368. $$\int \cos nx \cos^m x\,dx = \frac{1}{m+n}\Big[\sin nx \cos^m x + m\int \cos(n-1)x \cdot \cos^{m-1}x\,dx\Big].$$

369. $$\int \frac{\cos nx\,dx}{\cos^m x} = 2\int \frac{\cos(n-1)x\,dx}{\cos^{m-1}x} - \int \frac{\cos(n-2)x\,dx}{\cos^m x}.$$

370. $$\int \frac{\cos nx\,dx}{\sin^m x} = -2\int \frac{\sin(n-1)x\,dx}{\sin^{m-1}x} + \int \frac{\cos(n-2)x\,dx}{\sin^m x}.$$

371. $$\int \frac{\sin nx\,dx}{\sin^m x} = 2\int \frac{\cos(n-1)x\,dx}{\sin^{m-1}x} + \int \frac{\sin(n-2)x\,dx}{\sin^m x}.$$

372. $$\int \frac{\sin nx\,dx}{\cos^m x} = 2\int \frac{\sin(n-1)x\,dx}{\cos^{m-1}x} - \int \frac{\sin(n-2)x\,dx}{\cos^m x}.$$

373. $$\int \frac{(\cos px + i \sin px)\,dx}{\cos nx} = -2i\int \frac{z^{p+n-1}\,dz}{1+z^{2n}},$$

where $z = \cos x + i \sin x$. This yields two real integrals.

374. $$\int \frac{(\cos px + i \sin px)\,dx}{\sin nx} = -2\int \frac{z^{p+n-1}\,dz}{1-z^{2n}},$$

where $z = \cos x + i \sin x$. This yields two real integrals.

375. $\int \frac{(i\cos x - \sin x)\,dx}{\sqrt[n]{\cos nx}} = \int \frac{dy}{2 - y^n},$

where $y = \frac{\cos x + i\sin x}{\sqrt[n]{\cos nx}}$. This yields two real integrals.

376. $\int \sin ax \sin bx \sin cx\,dx = -\tfrac{1}{4}\left\{\frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c}\right\}.$

377. $\int \cos ax \cos bx \cos cx\,dx = \tfrac{1}{4}\left\{\frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(a-b+c)x}{a-b+c} + \frac{\sin(a+b-c)x}{a+b-c}\right\}.$

378. $\int \sin ax \cos bx \cos cx\,dx = -\tfrac{1}{4}\left\{\frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b}\right\}.$

379. $\int \cos ax \sin bx \sin cx\,dx = \tfrac{1}{4}\left\{\frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a-b+c)x}{a-b+c} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a}\right\}.$

380. $\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2}.$

381. $\int \cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1-x^2}.$

382. $\int \tan^{-1}x\,dx = x\tan^{-1}x - \tfrac{1}{2}\log(1+x^2).$

383. $\int \text{ctn}^{-1}x\,dx = x\,\text{ctn}^{-1}x + \tfrac{1}{2}\log(1+x^2).$

384. $\int \sec^{-1} x\,dx = x \sec^{-1} x - \log\left(x + \sqrt{x^2 + 1}\right).$

385. $\int \csc^{-1} x\,dx = x \csc^{-1} x + \log\left(x + \sqrt{x^2 + 1}\right).$

386. $\int \operatorname{versin}^{-1} x\,dx = (x - 1) \operatorname{versin}^{-1} x + \sqrt{2x - x^2}.$

387. $\int (\sin^{-1} x)^2 dx = x (\sin^{-1} x)^2 - 2x + 2\sqrt{1 - x^2} \sin^{-1} x.$

388. $\int (\cos^{-1} x)^2 dx = x (\cos^{-1} x)^2 - 2x - 2\sqrt{1 - x^2} \cos^{-1} x.$

389. $\int x \sin^{-1} x\,dx = \frac{1}{4}\left[(2x^2 - 1) \sin^{-1} x + x\sqrt{1 - x^2}\right].$

390. $\int x \cos^{-1} x\,dx = \frac{1}{4}\left[(2x^2 - 1) \cos^{-1} x - x\sqrt{1 - x^2}\right].$

391. $\int x \tan^{-1} x\,dx = \frac{1}{2}\left[(x^2 + 1) \tan^{-1} x - x\right].$

392. $\int x \operatorname{ctn}^{-1} x\,dx = \frac{1}{2}\left[(x^2 + 1) \operatorname{ctn}^{-1} x + x\right].$

393. $\int x \sec^{-1} x\,dx = \frac{1}{2}\left[x^2 \sec^{-1} x - \sqrt{x^2 - 1}\right].$

394. $\int x \csc^{-1} x\,dx = \frac{1}{2}\left[x^2 \csc^{-1} x + \sqrt{x^2 - 1}\right].$

395. $\int x^n \sin^{-1} x\,dx = \frac{1}{n + 1}\left(x^{n+1} \sin^{-1} x - \int \frac{x^{n+1}\,dx}{\sqrt{1 - x^2}}\right).$

396. $\int x^n \cos^{-1} x\,dx = \frac{1}{n + 1}\left(x^{n+1} \cos^{-1} x + \int \frac{x^{n+1}\,dx}{\sqrt{1 - x^2}}\right).$

397. $\int x^n \tan^{-1} x\,dx = \frac{1}{n+1}\left(x^{n+1}\tan^{-1}x - \int \frac{x^{n+1}\,dx}{1+x^2}\right).$

398. $\int x^n \operatorname{ctn}^{-1} x\,dx = \frac{1}{n+1}\left(x^{n+1}\operatorname{ctn}^{-1}x + \int \frac{x^{n+1}\,dx}{1+x^2}\right).$

399. $\int \frac{\sin^{-1}x\,dx}{x^2} = \log\left(\frac{1-\sqrt{1-x^2}}{x}\right) - \frac{\sin^{-1}x}{x}.$

400. $\int \frac{\tan^{-1}x\,dx}{x^2} = \log x - \tfrac{1}{2}\log(1+x^2) - \frac{\tan^{-1}x}{x}.$

401. $\int e^{ax}\,dx = \frac{e^{ax}}{a}.$

402. $\int x e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1).$

403. $\int x^m e^{ax}\,dx = \frac{x^m e^{ax}}{a} - \frac{m}{a}\int x^{m-1}e^{ax}\,dx.$

404. $\int \frac{e^{ax}}{x^m}\,dx = \frac{1}{m-1}\left[-\frac{e^{ax}}{x^{m-1}} + a\int \frac{e^{ax}\,dx}{x^{m-1}}\right].$

405. $\int a^{bx}\,dx = \frac{a^{bx}}{b\log a}.$

406. $$\int x^n a^x\,dx = \frac{a^x x^n}{\log a} - \frac{n a^x x^{n-1}}{(\log a)^2} + \frac{n(n-1)\,a^x x^{n-2}}{(\log a)^3} \cdots$$
$$\pm \frac{n(n-1)(n-2)\cdots 2.1\,a^x}{(\log a)^{n+1}}.$$

407. $$\int \frac{a^x\,dx}{x^n} = \frac{a^x}{n-1}\left[-\frac{1}{x^{n-1}} - \frac{\log a}{(n-2)\,x^{n-2}}\right.$$
$$\left. - \frac{(\log a)^2}{(n-2)(n-3)\,x^{n-3}} - \cdots + \frac{(\log a)^{n-1}}{(n-2)(n-3)\cdots 2.1}\int \frac{a^x\,dx}{x}\right].$$

408. $\int \frac{a^x\,dx}{x} = \log x + x\log a + \frac{(x\log a)^2}{2\cdot 2!} + \frac{(x\log a)^3}{3\cdot 3!} + \cdots.$

409. $$\int \frac{dx}{1+e^x} = \log \frac{e^x}{1+e^x}.$$

410. $$\int \frac{dx}{a+be^{mx}} = \frac{1}{am}[mx - \log(a+be^{mx})].$$

411. $$\int \frac{dx}{ae^{mx}+be^{-mx}} = \frac{1}{m\sqrt{ab}} \tan^{-1}\left(e^{mx}\sqrt{\frac{a}{b}}\right).$$

412. $$\int \frac{dx}{\sqrt{a+be^{mx}}} = \frac{1}{m\sqrt{a}}\{\log(\sqrt{a+be^{mx}} - \sqrt{a})$$
$$- \log(\sqrt{a+be^{mx}} + \sqrt{a})\}.$$

413. $$\int \frac{xe^x\,dx}{(1+x)^2} = \frac{e^x}{1+x}, \quad \int x^n \cdot e^{ax^{n+1}} dx = \frac{e^{ax^{n+1}}}{a(n+1)}.$$

414. $$\int e^{ax} \sin x\,dx = \frac{e^{ax}(a \sin x - \cos x)}{1+a^2}.$$

415. $$\int e^{ax} \cos x\,dx = \frac{e^{ax}(\sin x + a \cos x)}{1+a^2}.$$

416. $$\int e^{ax} \log x\,dx = \frac{e^{ax}\log x}{a} - \frac{1}{a}\int \frac{e^{ax}\,dx}{x}.$$

417. $$\int e^{ax} \sin^2 x\,dx = \frac{e^{ax}}{4+a^2}\left(\sin x(a \sin x - 2\cos x) + \frac{2}{a}\right).$$

418. $$\int e^{ax} \cos^2 x\,dx = \frac{e^{ax}}{4+a^2}\left(\cos x(2 \sin x + a \cos x) + \frac{2}{a}\right).$$

419. $$\int e^{ax} \sin^n bx\,dx = \frac{1}{a^2+n^2b^2}\Big((a \sin bx$$
$$- nb \cos bx)\, e^{ax} \sin^{n-1} bx + n(n-1)\,b^2 \int e^{ax} \sin^{n-2} bx \cdot dx\Big).$$

420. $$\int e^{ax}\cos^n bx\,dx = \frac{1}{a^2+n^2b^2}\Big((a\cos bx + nb\sin bx)\,e^{ax}\cos^{n-1}bx + n(n-1)\,b^2\int e^{ax}\cos^{n-2}bx\,dx\Big).$$

421. $$\int e^{ax}\tan^n x\,dx = \frac{e^{ax}\tan^{n-1}x}{n-1} - \frac{a}{n-1}\int e^{ax}\tan^{n-1}x\,dx - \int e^{ax}\tan^{n-2}x\,dx.$$

422. $$\int e^{ax}\operatorname{ctn}^n x\,dx = -\frac{e^{ax}\operatorname{ctn}^{n-1}x}{n-1} + \frac{a}{n-1}\int e^{ax}\operatorname{ctn}^{n-1}x\,dx - \int e^{ax}\operatorname{ctn}^{n-2}x\,dx.$$

423. $$\int\frac{e^{ax}\,dx}{\sin^n x} = -\,e^{ax}\frac{a\sin x + (n-2)\cos x}{(n-1)(n-2)\sin^{n-1}x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)}\int\frac{e^{ax}\,dx}{\sin^{n-2}x}.$$

424. $$\int\frac{e^{ax}\,dx}{\cos^n x} = -\,e^{ax}\frac{a\cos x - (n-2)\sin x}{(n-1)(n-2)\cos^{n-1}x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)}\int\frac{e^{ax}\,dx}{\cos^{n-2}x}.$$

425. $$\int e^{ax}\sin^m x\cos^n x\,dx = \frac{1}{(m+n)^2+a^2}\Big\{e^{ax}\sin^{mx}x\cos^{n-1}x\,(a\cos x + (m+n)\sin x) - ma\int e^{ax}\sin^{m-1}x\cos^{m-1}x\,dx + (n-1)(m+n)\int e^{ax}\sin^m x\cos^{n-2}x\,dx\Big\}$$

$$
\begin{aligned}
&= \frac{1}{(m+n)^2+a^2}\Big\{ e^{ax} \sin^{m-1} x \cos^n x (a \sin x - (m+n) \cos x) \\
&\qquad + na \int e^{ax} \sin^{m-1} x \cos^{n-1} x \, dx \\
&\qquad + (m-1)(m+n) \int e^{ax} \sin^{m-2} x \cos^n x \, dx \Big\} \\
&= \frac{1}{(m+n)^2+a^2}\Big\{ [e^{ax} \cos^{n-1} x \sin^{m-1} x (a \sin x \cos x + n \sin^2 x \\
&\qquad - m \cos^2 x)] + n(n-1) \int e^{ax} \sin^m x \cos^{n-2} x \, dx \\
&\qquad + m(m-1) \int e^{ax} \sin^{m-2} x \cos^n x \, dx \Big\} \\
&= \frac{1}{(m+n)^2+a^2}\Big\{ [e^{ax} \sin^{m-1} x \cos^{n-1} x (a \sin x \cos x + n \sin^2 x \\
&\qquad - m \cos^2 x)] + n(n-1) \int e^{ax} \sin^{m-2} x \cos^{n-2} x \, dx \\
&\qquad + (m-n)(m+n-1) \int e^{ax} \sin^{m-2} x \cos^n x \, dx \Big\} \\
&= \frac{1}{(m+n)^2+a^2}\Big\{ [e^{ax} \sin^{m-1} x \cos^{n-1} x (a \sin x \cos x + n \sin^2 x \\
&\qquad - m \cos^2 x)] + m(m-1) \int e^{ax} \sin^{m-2} x \cos^{n-2} x \, dx \\
&\qquad - (m-n)(m+n-1) \int e^{ax} \sin^m x \cos^{n-1} x \, dx \Big\}.
\end{aligned}
$$

426. $\int \log x \, dx = x \log x - x.$

427. $\int x^m \log x \, dx = x^{m+1} \left[\frac{\log x}{m+1} - \frac{1}{(m+1)^2} \right].$

428. $\int (\log x)^n dx = x (\log x)^n - n \int (\log x)^{n-1} dx.$

429. $\int x^m (\log x)^n dx = \frac{x^{m+1} (\log x)^n}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx.$

430. $$\int \frac{(\log x)^n\,dx}{x} = \frac{(\log x)^{n+1}}{n+1}.$$

431. $$\int \frac{dx}{\log x} = \log(\log x) + \log x + \frac{(\log x)^2}{2\cdot 2!} + \frac{(\log x)^3}{3\cdot 3!} + \cdots.$$

432. $$\int \frac{dx}{(\log x)^n} = -\frac{x}{(n-1)(\log x)^{n-1}} + \frac{1}{n-1}\int \frac{dx}{(\log x)^{n-1}}.$$

433. $$\int \frac{x^m\,dx}{(\log x)^n} = -\frac{x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m\,dx}{(\log x)^{n-1}}.$$

434. $$\int \frac{x^m\,dx}{\log x} = \int \frac{e^{-y}}{y}\,dy, \text{ where } y = -(m+1)\log x.$$

435. $$\int \frac{dx}{x\log x} = \log(\log x), \text{ and } \int \frac{(n-1)\,dx}{x(\log x)^n} = \frac{-1}{(\log x)^{n-1}}.$$

436. $$\int \log(a^2+x^2)\,dx = x\cdot\log(a^2+x^2) - 2x + 2a\cdot\tan^{-1}\left(\frac{x}{a}\right).$$

437. $$\int (a+bx)^m \log x\,dx$$
$$= \frac{1}{b(m+1)}\left[(a+bx)^{m+1}\log x - \int \frac{(a+bx)^{m+1}\,dx}{x}\right].$$

438. $$\int x^m \log(a+bx)\,dx$$
$$= \frac{1}{m+1}\left[x^{m+1}\log(a+bx) - b\int \frac{x^{m+1}\,dx}{a+bx}\right].$$

439. $$\int \frac{\log(a+bx)\,dx}{x}$$
$$= \log a\cdot\log x + \frac{bx}{a} - \frac{1}{2^2}\left(\frac{bx}{a}\right)^2 + \frac{1}{3^2}\left(\frac{bx}{a}\right)^3 - \cdots$$
$$= \tfrac{1}{2}(\log bx)^2 - \frac{a}{bx} + \frac{1}{2^2}\left(\frac{a}{bx}\right)^2 - \frac{1}{3^2}\left(\frac{a}{bx}\right)^3 + \cdots.$$

440. $$\int\frac{\log x\,dx}{(a+bx)^m}$$
$$=\frac{1}{b(m-1)}\left[-\frac{\log x}{(a+bx)^{m-1}}+\int\frac{dx}{x(a+bx)^{m-1}}\right].$$

441. $$\int\frac{\log x\,dx}{a+bx}=\frac{1}{b}\log x\cdot\log(a+bx)-\frac{1}{b}\int\frac{\log(a+bx)\,dx}{x}.$$

442. $$\int(a+bx)\log x\,dx=\frac{(a+bx)^2}{2b}\log x-\frac{a^2\log x}{2b}-ax-\tfrac{1}{4}bx^2.$$

443. $$\int\frac{\log x\,dx}{\sqrt{a+bx}}$$
$$=\frac{2}{b}\Big[(\log x-2)\sqrt{a+bx}+\sqrt{a}\log(\sqrt{a+bx}+\sqrt{a})$$
$$-\sqrt{a}\log(\sqrt{a+bx}-\sqrt{a})\Big],\text{ if } a>0$$
$$=\frac{2}{b}\left[(\log x-2)\sqrt{a+bx}+2\sqrt{-a}\tan^{-1}\sqrt{\frac{a+bx}{-a}}\right],\text{ if } a<0.$$

444. $$\int\sin\log x\,dx=\tfrac{1}{2}x[\sin\log x-\cos\log x].$$

445. $$\int\cos\log x\,dx=\tfrac{1}{2}x[\sin\log x+\cos\log x].$$

446. $$\int\sinh x\,dx=\cosh x.$$

447. $$\int\cosh x\,dx=\sinh x.$$

448. $$\int\tanh x\,dx=\log\cosh x.$$

449. $$\int\operatorname{ctnh} x\,dx=\log\sinh x.$$

450. $\int \operatorname{sech} x\,dx = 2\tan^{-1} e^x.$

451. $\int \operatorname{csch} x\,dx = \log\tanh\frac{x}{2}\cdot$

452. $$\int \sinh^n x\,dx = \frac{1}{n}\sinh^{n-1}x\cdot\cosh x - \frac{n-1}{n}\int\sinh^{n-2}x\,dx$$
$$= \frac{1}{n+1}\sinh^{n+1}x\cosh x - \frac{n+2}{n+1}\int\sinh^{n+2}x\,dx.$$

453. $$\int \cosh^n x\,dx = \frac{1}{n}\sinh x\cdot\cosh^{n-1}x + \frac{n-1}{n}\int\cosh^{n-2}x\,dx$$
$$= -\frac{1}{n+1}\sinh x\cosh^{n+1}x + \frac{n+2}{n+1}\int\cosh^{n+2}x\,dx.$$

454. $\int x\sinh x\,dx = x\cosh x - \sinh x.$

455. $\int x\cosh x\,dx = x\sinh x - \cosh x.$

456. $\int x^2\sinh x\,dx = (x^2+2)\cosh x - 2x\sinh x.$

457. $$\int x^n\sinh x\,dx = x^n\cosh x - nx^{n-1}\sinh x$$
$$+ n(n-1)\int x^{n-2}\sinh x\,dx.$$

458. $\int \sinh^2 x\,dx = \frac{1}{2}(\sinh x\cosh x - x).$

459. $\int \sinh x\cdot\cosh x\,dx = \frac{1}{4}\cosh(2x).$

460. $\int \cosh^2 x\,dx = \frac{1}{2}(\sinh x\cosh x + x).$

461. $\int \tanh^2 x\,dx = x - \tanh x.$

462. $\int \operatorname{ctnh}^2 x\,dx = x - \operatorname{ctnh} x.$

463. $\int \operatorname{sech}^2 x\,dx = \tanh x.$

464. $\int \operatorname{csch}^2 x\,dx = -\operatorname{ctnh} x.$

465. $\int \sinh^{-1} x\,dx = x \sinh^{-1} x - \sqrt{1 + x^2}.$

466. $\int \cosh^{-1} x\,dx = x \cosh^{-1} x - \sqrt{x^2 - 1}.$

467. $\int \tanh^{-1} x\,dx = x \tanh^{-1} x + \frac{1}{2} \log (1 - x^2).$

468. $\int x \sinh^{-1} x\,dx = \frac{1}{4}[(2x^2 + 1) \sinh^{-1} x - x\sqrt{1 + x^2}].$

469. $\int x \cosh^{-1} x\,dx = \frac{1}{4}[(2x^2 - 1) \cosh^{-1} x - x\sqrt{x^2 - 1}].$

470. $\int \frac{dx}{\cosh a + \cosh x}$

$= \operatorname{csch} a [\log \cosh \frac{1}{2}(x + a) - \log \cosh \frac{1}{2}(x - a)],$

$= 2 \operatorname{csch} a \cdot \tanh^{-1}(\tanh \frac{1}{2} x \cdot \tanh \frac{1}{2} a).$

471. $\int \frac{dx}{\cos a + \cosh x} = 2 \csc a \cdot \tan^{-1}(\tanh \frac{1}{2} x \cdot \tan \frac{1}{2} a).$

472. $\int \frac{dx}{1 + \cos a \cdot \cosh x} = 2 \csc a \cdot \tanh^{-1}(\tanh \frac{1}{2} x \cdot \tan \frac{1}{2} a).$

473. $\int \sinh x \cdot \cos x\,dx = \frac{1}{2}(\cosh x \cdot \cos x + \sinh x \cdot \sin x).$

474. $\int \cosh x \cdot \cos x\,dx = \frac{1}{2}(\sinh x \cdot \cos x + \cosh x \cdot \sin x).$

475. $\int \sinh x \cdot \sin x\,dx = \frac{1}{2}(\cosh x \cdot \sin x - \sinh x \cdot \cos x).$

476. $\int \cosh x \cdot \sin x\,dx = \frac{1}{2}(\sinh x \cdot \sin x - \cosh x \cdot \cos x).$

477. $\int \sinh(mx)\sinh(nx)\,dx$

$$= \frac{1}{m^2 - n^2}\Big[m \sinh(nx)\cosh(mx) - n\cosh(nx)\sinh(mx)\Big].$$

478. $\int \cosh(mx)\sinh(nx)\,dx$

$$= \frac{1}{m^2 - n^2}\Big[m \sinh(nx)\sinh(mx) - n\cosh(nx)\cosh(mx)\Big].$$

479. $\int \cosh(mx)\cosh(nx)\,dx$

$$= \frac{1}{m^2 - n^2}\Big[m \sinh(mx)\cosh(nx) - n\sinh(nx)\cosh(mx)\Big].$$

VI. MISCELLANEOUS DEFINITE INTEGRALS.*

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480. $\int_0^\infty \frac{a\,dx}{a^2+x^2} = \frac{\pi}{2}$, if $a > 0$; 0, if $a = 0$; $-\frac{\pi}{2}$, if $a < 0$.

481. $\int_0^\infty x^{n-1} e^{-x}\,dx = \int_0^1 \left[\log\frac{1}{x}\right]^{n-1} dx \equiv \Gamma(n)$.

$$\Gamma(z+1) = z\cdot\Gamma(z), \text{ if } z>0.$$

$\Gamma(y)\cdot\Gamma(1-y) = \frac{\pi}{\sin \pi y}$, if $1 > y > 0$. $\Gamma(2) = \Gamma(1) = 1$.

$\Gamma(n+1) = n!$, if n is an integer. $\Gamma(z) = \Pi(z-1)$.

$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$. $Z(y) = D_y[\log\Gamma(y)]$. $Z(1) = -0.577216$.

482. $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

483. $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$

$= \frac{1\cdot3\cdot5\cdots(n-1)}{2\cdot4\cdot6\cdots(n)}\cdot\frac{\pi}{2}$, if n is an even integer,

$= \frac{2\cdot4\cdot6\cdots(n-1)}{1\cdot3\cdot5\cdot7\cdots n}$, if n is an odd integer,

$= \frac{1}{2}\sqrt{\pi}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$, for any value of n greater than -1.

484. $\int_0^\infty \frac{\sin mx\,dx}{x} = \frac{\pi}{2}$, if $m > 0$; 0, if $m = 0$; $-\frac{\pi}{2}$, if $m < 0$.

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* For very complete lists of definite integrals, see Bierens de Haan, *Tables d'intégrales définies*, Amsterdam, 1858-64, and *Nouv. Tables d'intégrales définies*, Leyden, 1867.

485. $\int_0^\infty \frac{\sin x \cdot \cos mx\, dx}{x} = 0$, if $m < -1$ or $m > 1$;

$\frac{\pi}{4}$, if $m = -1$ or $m = 1$; $\frac{\pi}{2}$, if $-1 < m < 1$.

486. $\int_0^\infty \frac{\sin^2 x\, dx}{x^2} = \frac{\pi}{2}$.

487. $\int_0^\infty \cos(x^2)\, dx = \int_0^\infty \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

488. $\int_0^\pi \sin kx \cdot \sin mx\, dx = \int_0^\pi \cos kx \cdot \cos mx\, dx = 0$,

if k is different from m.

489. $\int_0^\pi \sin^2 mx\, dx = \int_0^\pi \cos^2 mx\, dx = \frac{\pi}{2}$.

490. $\int_0^\infty \frac{\cos mx\, dx}{1 + x^2} = \frac{\pi}{2} \cdot e^{-m}$. $m > 0$.

491. $\int_0^\infty \frac{\cos x\, dx}{\sqrt{x}} = \int_0^\infty \frac{\sin x\, dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}$.

492. $\int_0^\infty e^{-a^2x^2}\, dx = \frac{1}{2a}\sqrt{\pi} = \frac{1}{2a}\Gamma(\frac{1}{2})$.

493. $\int_0^\infty x^n e^{-ax}\, dx = \frac{\Gamma(n+1)}{a^{n+1}} = \frac{n!}{a^{n+1}}$.

494. $\int_0^\infty x^{2n} e^{-ax^2}\, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n}\sqrt{\frac{\pi}{a}}$.

495. $\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\, dx = \frac{e^{-2a}\sqrt{\pi}}{2}$. $a > 0$.

496. $\int_0^\infty e^{-nx}\sqrt{x}\, dx = \frac{1}{2n}\sqrt{\frac{\pi}{n}}$.

497. $\int_0^\infty \frac{e^{-nx}}{\sqrt{x}}\, dx = \sqrt{\frac{\pi}{n}}$. $a > 0$.

498. $\int_0^\infty \frac{dx}{e^{nx}+e^{-nx}} = \frac{\pi}{4n}.$

499. $\int_0^\infty \frac{x\,dx}{e^{nx}-e^{-nx}} = \frac{\pi^2}{8n^2}.$

500. $\int_0^{\pi i} \sinh(mx)\cdot\sinh(nx)\,dx = \int_0^{\pi i} \cosh(mx)\cdot\cosh(nx)\,dx$

$= 0$, if m is different from n.

501. $\int_0^{\pi i} \cosh^2(mx)\,dx = -\int_0^{\pi i} \sinh^2(mx)\,dx = \frac{\pi i}{2}.$

502. $\int_{-\pi i}^{+\pi i} \sinh(mx)\,dx = 0.$

503. $\int_0^{\pi i} \cosh(mx)\,dx = 0.$

504. $\int_{-\pi i}^{\pi i} \sinh(mx)\cosh(nx)\,dx = 0.$

505. $\int_0^{\pi i} \sinh(mx)\cosh(mx)\,dx = 0.$

506. $\int_0^\infty e^{-ax}\cos mx\,dx = \frac{a}{a^2+m^2}$, if $a > 0$.

507. $\int_0^\infty e^{-ax}\sin mx\,dx = \frac{m}{a^2+m^2}$, if $a > 0$.

508. $\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}\cdot e^{-\frac{b^2}{4a^2}}}{2a}.$ $a > 0.$

509. $\int_0^1 \frac{\log x}{1-x}\,dx = -\frac{\pi^2}{6}.$

510. $\int_0^1 \frac{\log x}{1+x}\,dx = -\frac{\pi^2}{12}.$

511. $\int_0^1 \frac{\log x}{1-x^2}\,dx = -\frac{\pi^2}{8}.$

512. $\int_0^1 \log\left(\frac{1+x}{1-x}\right)\cdot\frac{dx}{x} = \frac{\pi^2}{4}.$

513. $\int_0^1 \frac{\log x\, dx}{\sqrt{1-x^2}} = -\frac{\pi}{2}\log 2.$

514. $\int_0^1 \frac{(x^p - x^q)\, dx}{\log x} = \log\frac{p+1}{q+1}$, if $p+1>0$, $q+1>0$.

515. $\int_0^1 (\log x)^n dx = (-1)^n \cdot n!.$

516. $\int_0^1 \left(\log\frac{1}{x}\right)^{\frac{1}{2}} dx = \frac{\sqrt{\pi}}{2}.$

517. $\int_0^1 \left(\log\frac{1}{x}\right)^n dx = n!.$

518. $\int_0^1 \frac{dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \sqrt{\pi}.$

519. $\int_0^1 x^m \log\left(\frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$, if $m+1>0$, $n+1>0$.

520. $\int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right) dx = \frac{\pi^2}{4}.$

521. $\int_0^{\frac{\pi}{2}} \log\sin x\, dx = \int_0^{\frac{\pi}{2}} \log\cos x\, dx = -\frac{\pi}{2}\cdot\log 2.$

522. $\int_0^\pi x\cdot\log\sin x\, dx = -\frac{\pi^2}{2}\log 2.$

523. $\int_0^\pi \log(a \pm b\cos x)\, dx = \pi\log\left(\frac{a+\sqrt{a^2-b^2}}{2}\right).\qquad a \geqq b.$

VII. ELLIPTIC INTEGRALS.

$$F(\phi, k) \equiv \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \equiv \int_0^{x} \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} \equiv u,$$

where $k^2 < 1$, $x = \sin\phi$.

$$E(\phi, k) \equiv \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\cdot d\theta.$$

$$\Pi(\phi, n, k) \equiv \int_0^{\phi} \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}.$$

$$\phi \equiv \operatorname{am} u,\ \sin\phi \equiv x \equiv \operatorname{sn} u,\ \cos\phi \equiv \sqrt{1-x^2} \equiv \operatorname{cn} u,\ \tan\phi \equiv \operatorname{tn} u,$$

$$\Delta\phi \equiv \sqrt{1-k^2\sin^2\phi} \equiv \sqrt{1-k^2x^2} \equiv \operatorname{dn} u,\ k'^2 \equiv 1-k^2.$$

$$u \equiv \operatorname{am}^{-1}(\phi, k) \equiv \operatorname{sn}^{-1}(x, k) \equiv \operatorname{cn}^{-1}(\sqrt{1-x^2}, k)$$
$$\equiv \operatorname{dn}^{-1}(\sqrt{1-k^2x^2}, k).$$

$$K \equiv F(\tfrac{1}{2}\pi, k),\ K' \equiv F(\tfrac{1}{2}\pi, k'),\ E \equiv E(\tfrac{1}{2}\pi, k),\ E' \equiv E(\tfrac{1}{2}\pi, k').$$

If $k_0 = \dfrac{2k^{\frac{1}{2}}}{1+k}$ and $\tan\phi \equiv \dfrac{\sin 2\omega}{k+\cos 2\omega}$,

$$F(\phi, k) \equiv \frac{2}{1+k} F(\omega, k_0).$$

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524. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$

$$= \frac{\pi}{2}\left[1 + (\tfrac{1}{2})^2k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right], \text{ if } k^2<1,$$

$= K.$

525. $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\cdot d\theta$

$$= \frac{\pi}{2}\left[1 - (\tfrac{1}{2})^2k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5} - \cdots\right], \text{ if } k^2<1,$$

$= E.$

526. $$\int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{2}{\pi}\phi\cdot K - \sin\phi\cos\phi\left[\frac{1\cdot 1}{2\cdot 2}k^2 + \frac{1\cdot 3}{2\cdot 4}A_4k^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}A_6k^6 + \cdots\right] = F(\phi, k),$$

where $A_4 \equiv \frac{1}{4}\sin^2\phi + \frac{3}{2\cdot 4}$, $A_6 \equiv \frac{1}{6}\sin^4\phi + \frac{5}{6\cdot 4}\sin^2\phi + \frac{5\cdot 3}{6\cdot 4\cdot 2}$, $A_8 \equiv \frac{1}{8}\sin^6\phi + \frac{7}{8\cdot 6}\sin^4\phi + \frac{7\cdot 5}{8\cdot 6\cdot 4}\sin^2\phi + \frac{7\cdot 5\cdot 3}{8\cdot 6\cdot 4\cdot 2}$, etc.

527. $$\int_0^\phi \sqrt{1-k^2\sin^2\theta}\cdot d\theta = \frac{2}{\pi}\phi\cdot E + \sin\phi\cos\phi\left[\frac{1\cdot 1}{2\cdot 2}k^2 + \frac{1}{2\cdot 4}k^4A_4 + \frac{1\cdot 3}{2\cdot 4\cdot 6}k^6A_6 + \cdots\right] = E(\phi, k).$$

528.* $$\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \text{sn}^{-1}(x, k) = F(\sin^{-1}x, k). \qquad 0 < x < 1.$$

529. $$\int_x^1 \frac{dx}{\sqrt{(1-x^2)(k'^2+k^2x^2)}} = \text{cn}^{-1}(x, k) = F(\cos^{-1}x, k) = \text{sn}^{-1}(\sqrt{1-x^2}, k). \qquad 0 < x < 1.$$

530. $$\int_x^1 \frac{dx}{\sqrt{(1-x^2)(x^2-k'^2)}} = \text{dn}^{-1}(x, k) = F(\Delta^{-1}x, k) = \text{sn}^{-1}\left(\frac{1}{k}\sqrt{1-x^2}, k\right)\cdot \qquad 0 < x < 1.$$

531. $$\int_0^x \frac{dx}{\sqrt{(1+x^2)(1+k'^2x^2)}} = \text{tn}^{-1}(x, k) = F(\tan^{-1}x, k) = \text{sn}^{-1}\left(\frac{x}{\sqrt{1+x^2}}, k\right)\cdot \qquad 0 < x < 1.$$

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* The next forty-two integrals are copied in order from a class-room list of Prof. W. E. Byerly.

532. $$\int_0^x \frac{dx}{\sqrt{x(1-x)(1-k^2x)}} = 2\,\mathrm{sn}^{-1}(\sqrt{x},\,k)$$
$$= 2\,F(\sin^{-1}\sqrt{x},\,k). \quad 0<x<1.$$

533. $$\int_x^1 \frac{dx}{\sqrt{x(1-x)(k'^2+k^2x)}} = 2\,\mathrm{cn}^{-1}(\sqrt{x},\,k)$$
$$= 2\,F(\cos^{-1}\sqrt{x},\,k) = 2\,\mathrm{sn}^{-1}(\sqrt{1-x},\,k). \quad 0<x<1.$$

534. $$\int_x^1 \frac{dx}{\sqrt{x(1-x)(x-k'^2)}} = 2\,\mathrm{dn}^{-1}(\sqrt{x},\,k)$$
$$= 2\,F(\Delta^{-1}\sqrt{x},\,k) = 2\,\mathrm{sn}^{-1}\left(\frac{1}{k}\sqrt{1-x},\,k\right). \quad 0<x<1.$$

535. $$\int_0^x \frac{dx}{\sqrt{(1+x)(1+k'^2x)}} = 2\,\mathrm{tn}^{-1}(\sqrt{x},\,k)$$
$$= 2\,F(\tan^{-1}\sqrt{x},\,k) = 2\,\mathrm{sn}^{-1}\left(\sqrt{\frac{x}{1+x}},\,k\right). \quad 0<x<1.$$

536. $$\int_0^x \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a}\,\mathrm{sn}^{-1}\left(\frac{x}{b},\,\frac{b}{a}\right). \qquad a>b>x>0.$$

537. $$\int_x^\infty \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a}\,\mathrm{sn}^{-1}\left(\frac{a}{x},\,\frac{b}{a}\right). \qquad x>a>b.$$

538. $$\int_x^b \frac{dx}{\sqrt{(a^2+x^2)(b^2-x^2)}}$$
$$= \frac{1}{\sqrt{a^2+b^2}}\,\mathrm{cn}^{-1}\left(\frac{x}{b},\,\frac{b}{\sqrt{a^2+b^2}}\right). \qquad b>x>0.$$

539. $$\int_b^x \frac{dx}{\sqrt{(a^2+x^2)(x^2-b^2)}}$$
$$= \frac{1}{\sqrt{a^2+b^2}}\,\mathrm{cn}^{-1}\left(\frac{b}{x},\,\frac{a}{\sqrt{a^2+b^2}}\right). \qquad x>b>0.$$

540. $$\int_x^a \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{a^2-x^2}{a^2-b^2}},\ \sqrt{\frac{a^2-b^2}{a^2}}\right). \qquad a > x > b.$$

541. $$\int_0^x \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a}\,\mathrm{tn}^{-1}\left(\frac{x}{b},\ \sqrt{\frac{a^2-b^2}{a^2}}\right). \qquad x > 0.$$

$\alpha > \beta > \gamma.$

542. $$\int_x^\infty \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-\gamma}{x-\gamma}},\ \sqrt{\frac{\beta-\gamma}{\alpha-\gamma}}\right). \qquad x > \alpha.$$

543. $$\int_\alpha^x \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{x-\alpha}{x-\beta}},\ \sqrt{\frac{\beta-\gamma}{\alpha-\gamma}}\right). \qquad x > \alpha.$$

544. $$\int_x^\alpha \frac{dx}{\sqrt{(\alpha-x)(x-\beta)(x-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-x}{\alpha-\beta}},\ \sqrt{\frac{\alpha-\beta}{\alpha-\gamma}}\right). \qquad \alpha > x > \beta.$$

545. $$\int_\beta^x \frac{dx}{\sqrt{(\alpha-x)(x-\beta)(x-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-\gamma}{\alpha-\beta}\cdot\frac{x-\beta}{x-\gamma}},\ \sqrt{\frac{\alpha-\beta}{\alpha-\gamma}}\right). \qquad \alpha > x > \beta.$$

546. $$\int_x^\beta \frac{dx}{\sqrt{(\alpha-x)(\beta-x)(x-\gamma)}} = \frac{2}{\sqrt{\alpha-\gamma}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-\gamma}{\beta-\gamma}\cdot\frac{\beta-x}{\alpha-x}},\ \sqrt{\frac{\beta-\gamma}{\alpha-\gamma}}\right). \qquad \beta > x > \gamma.$$

547. $$\int_{\gamma}^{x}\frac{dx}{\sqrt{(\alpha-x)(\beta-x)(x-\gamma)}}$$
$$=\frac{2}{\sqrt{\alpha-\gamma}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{x-\gamma}{\beta-\gamma}},\ \sqrt{\frac{\beta-\gamma}{\alpha-\gamma}}\right).\qquad \beta>x>\gamma.$$

548. $$\int_{x}^{\gamma}\frac{dx}{\sqrt{(\alpha-x)(\beta-x)(\gamma-x)}}$$
$$=\frac{2}{\sqrt{\alpha-\gamma}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\gamma-x}{\beta-x}},\ \sqrt{\frac{\alpha-\beta}{\alpha-\gamma}}\right).\qquad \gamma>x.$$

549. $$\int_{-\infty}^{x}\frac{dx}{\sqrt{(\alpha-x)(\beta-x)(\gamma-x)}}$$
$$=\frac{2}{\sqrt{\alpha-\gamma}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-\gamma}{\alpha-x}},\ \sqrt{\frac{\alpha-\beta}{\alpha-\gamma}}\right).\qquad \gamma>x.$$

$$\alpha>\beta>\gamma>\delta.$$

550. $$\int_{\alpha}^{x}\frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}}$$
$$=\frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\beta-\delta}{\alpha-\delta}\cdot\frac{x-\alpha}{x-\beta}},\ \sqrt{\frac{\beta-\gamma}{\alpha-\gamma}\cdot\frac{\alpha-\delta}{\beta-\delta}}\right).$$
$$x>\alpha.$$

551. $$\int_{x}^{\alpha}\frac{dx}{\sqrt{(\alpha-x)(x-\beta)(x-\gamma)(x-\delta)}}$$
$$=\frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\beta-\delta}{\alpha-\beta}\cdot\frac{\alpha-x}{x-\delta}},\ \sqrt{\frac{\alpha-\beta}{\alpha-\gamma}\cdot\frac{\gamma-\delta}{\beta-\delta}}\right).$$
$$\alpha>x>\beta.$$

552. $$\int_{\beta}^{x}\frac{dx}{\sqrt{(\alpha-x)(x-\beta)(x-\gamma)(x-\delta)}}$$
$$=\frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-\gamma}{\alpha-\beta}\cdot\frac{x-\beta}{x-\gamma}},\ \sqrt{\frac{\alpha-\beta}{\alpha-\gamma}\cdot\frac{\gamma-\delta}{\beta-\delta}}\right).$$
$$\alpha>x>\beta.$$

553. $$\int_x^\beta \frac{dx}{\sqrt{(\alpha-x)(\beta-x)(x-\gamma)(x-\delta)}}$$
$$=\frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-\gamma}{\beta-\gamma}\cdot\frac{\beta-x}{\alpha-x}},\ \sqrt{\frac{\beta-\gamma}{\alpha-\gamma}\cdot\frac{\alpha-\delta}{\beta-\delta}}\right).$$
$\beta > x > \gamma.$

554. $$\int_\gamma^x \frac{dx}{\sqrt{(\alpha-x)(\beta-x)(x-\gamma)(x-\delta)}}$$
$$=\frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\beta-\delta}{\beta-\gamma}\cdot\frac{x-\gamma}{x-\delta}},\ \sqrt{\frac{\beta-\gamma}{\alpha-\gamma}\cdot\frac{\alpha-\delta}{\beta-\delta}}\right).$$
$\beta > x > \gamma.$

555. $$\int_x^\gamma \frac{dx}{\sqrt{(\alpha-x)(\beta-x)(\gamma-x)(x-\delta)}}$$
$$=\frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\beta-\delta}{\gamma-\delta}\cdot\frac{\gamma-x}{\beta-x}},\ \sqrt{\frac{\alpha-\beta}{\alpha-\gamma}\cdot\frac{\gamma-\delta}{\beta-\delta}}\right).$$
$\gamma > x > \delta.$

556. $$\int_\delta^x \frac{dx}{\sqrt{(\alpha-x)(\beta-x)(\gamma-x)(x-\delta)}}$$
$$=\frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-\gamma}{\gamma-\delta}\cdot\frac{x-\delta}{\alpha-x}},\ \sqrt{\frac{\alpha-\beta}{\alpha-\gamma}\cdot\frac{\gamma-\delta}{\beta-\delta}}\right).$$
$\gamma > x > \delta.$

557. $$\int_x^\delta \frac{dx}{\sqrt{(\alpha-x)(\beta-x)(\gamma-x)(\delta-x)}}$$
$$=\frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\alpha-\gamma}{\alpha-\delta}\cdot\frac{\delta-x}{\gamma-x}},\ \sqrt{\frac{\beta-\gamma}{\alpha-\gamma}\cdot\frac{\alpha-\delta}{\beta-\delta}}\right).$$
$\delta > x.$

558. $$\int \mathrm{sn}\,x\,dx=\frac{1}{k}\cosh^{-1}\left(\frac{\mathrm{dn}\,x}{k'}\right).$$

559. $$\int \mathrm{cn}\,x\,dx=\frac{1}{k}\cos^{-1}(\mathrm{dn}\,x).$$

560. $$\int \operatorname{dn} x\, dx = \sin^{-1}(\operatorname{sn} x) = \operatorname{am} x.$$

561. $$\int \frac{dx}{\operatorname{sn} x} = \log\left[\frac{\operatorname{sn} x}{\operatorname{cn} x + \operatorname{dn} x}\right].$$

562. $$\int \frac{dx}{\operatorname{cn} x} = \frac{1}{k'} \log\left[\frac{k' \operatorname{sn} x + \operatorname{dn} x}{\operatorname{cn} x}\right].$$

563. $$\int \frac{dx}{\operatorname{dn} x} = \frac{1}{k'} \tan^{-1}\left[\frac{k' \operatorname{sn} x - \operatorname{cn} x}{k' \operatorname{sn} x + \operatorname{cn} x}\right].$$

564. $$\int_0^x \operatorname{sn}^2 x\, dx = \frac{1}{k^2}[x - E(\operatorname{am} x,\, k)].$$

565. $$\int_0^x \operatorname{cn}^2 x\, dx = \frac{1}{k^2}[E(\operatorname{am} x,\, k) - k'^2 x].$$

566. $$\int_0^x \operatorname{dn}^2 x\, dx = E(\operatorname{am} x,\, k).$$

567. $$(m+1)\int \operatorname{sn}^m x\, dx = (m+2)(1+k^2)\int \operatorname{sn}^{m+2} x\, dx - (m+3)k^2\int \operatorname{sn}^{m+4} x\, dx + \operatorname{sn}^{m+1} x \operatorname{cn} x \operatorname{dn} x.$$

568. $$(m+1)k'^2\int \operatorname{cn}^m x\, dx = (m+2)(1-2k^2)\int \operatorname{cn}^{m+2} x\, dx + (m+3)k^2\int \operatorname{cn}^{m+4} x\, dx - \operatorname{cn}^{m+1} x \operatorname{sn} x \operatorname{dn} x.$$

569. $$(m+1)k'^2\int \operatorname{dn}^m x\, dx = (m+2)(2-k^2)\int \operatorname{dn}^{m+2} x\, dx - (m+3)\int \operatorname{dn}^{m+4} x\, dx + k^2 \operatorname{dn}^{m+1} x \operatorname{sn} x \operatorname{cn} x.$$

Since $\sin^2\theta \equiv \frac{1}{k^2} - \frac{1}{k^2}(1 - k^2 \cdot \sin^2\theta)$,

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \cdot d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{1}{k^2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} - \frac{1}{k^2}\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta} \cdot d\theta.$$

VIII. AUXILIARY FORMULAS.

A.—TRIGONOMETRIC FUNCTIONS.

570. $\tan a \cdot \operatorname{ctn} a = \sin a \cdot \csc a = \cos a \cdot \sec a = 1,$

$$\tan a = \sin a \div \cos a, \quad \sec^2 a = 1 + \tan^2 a,$$

$$\csc^2 a = 1 + \operatorname{ctn}^2 a, \quad \sin^2 a + \cos^2 a = 1.$$

571. $\sin a = \sqrt{1-\cos^2 a} = 2\sin\tfrac{1}{2}a \cdot \cos\tfrac{1}{2}a = \cos a \cdot \tan a$

$$= \frac{1}{\sqrt{1+\operatorname{ctn}^2 a}} = \frac{\tan a}{\sqrt{1+\tan^2 a}} = \sqrt{\frac{1-\cos 2a}{2}} = \frac{2\tan\frac{1}{2}a}{1+\tan^2\frac{1}{2}a}$$

$$= \sqrt{\frac{\sec^2 a - 1}{\sec^2 a}} = \operatorname{ctn}\tfrac{1}{2}a \cdot (1-\cos a) = \tan\tfrac{1}{2}a \cdot (1+\cos a).$$

572. $\cos a = \sqrt{1-\sin^2 a} = \dfrac{1}{\sqrt{1+\tan^2 a}} = \dfrac{\operatorname{ctn} a}{\sqrt{1+\operatorname{ctn}^2 a}}$

$$= \sqrt{\frac{1+\cos 2a}{2}} = \frac{1-\tan^2\frac{1}{2}a}{1+\tan^2\frac{1}{2}a} = \cos^2\tfrac{1}{2}a - \sin^2\tfrac{1}{2}a$$

$$= 1 - 2\sin^2\tfrac{1}{2}a = 2\cos^2\tfrac{1}{2}a - 1 = \sin a \cdot \operatorname{ctn} a$$

$$= \frac{\sin 2a}{2\sin a} = \sqrt{\frac{\csc^2 a - 1}{\csc^2 a}} = \frac{\operatorname{ctn}\frac{1}{2}a - \tan\frac{1}{2}a}{\operatorname{ctn}\frac{1}{2}a + \tan\frac{1}{2}a}.$$

573. $\tan a = \dfrac{\sin a}{\sqrt{1-\sin^2 a}} = \dfrac{\sqrt{1-\cos^2 a}}{\cos a} = \dfrac{\sin 2a}{1+\cos 2a}$

$$= \frac{1-\cos 2a}{\sin 2a} = \sqrt{\frac{1-\cos 2a}{1+\cos 2a}} = \frac{2\tan\frac{1}{2}a}{1-\tan^2\frac{1}{2}a}$$

$$= \frac{\sec a}{\csc a} = \frac{2}{\operatorname{ctn}\frac{1}{2}a - \tan\frac{1}{2}a} = \frac{2\operatorname{ctn}\frac{1}{2}a}{\operatorname{ctn}^2\frac{1}{2}a - 1}.$$

574.

	$-\alpha$.	$90^\circ \pm \alpha$.	$180^\circ \pm \alpha$.	$270^\circ \pm \alpha$.	$360^\circ \pm \alpha$.
sin	$-\sin\alpha$	$+\cos\alpha$	$\mp\sin\alpha$	$-\cos\alpha$	$\pm\sin\alpha$
cos	$+\cos\alpha$	$\mp\sin\alpha$	$-\cos\alpha$	$\pm\sin\alpha$	$+\cos\alpha$
tan	$-\tan\alpha$	$\mp\operatorname{ctn}\alpha$	$\pm\tan\alpha$	$\mp\operatorname{ctn}\alpha$	$\pm\tan\alpha$
ctn	$-\operatorname{ctn}\alpha$	$\mp\tan\alpha$	$\pm\operatorname{ctn}\alpha$	$\mp\tan\alpha$	$\pm\operatorname{ctn}\alpha$
sec	$+\sec\alpha$	$\mp\csc\alpha$	$-\sec\alpha$	$\pm\csc\alpha$	$+\sec\alpha$
csc	$-\csc\alpha$	$+\sec\alpha$	$\mp\csc\alpha$	$-\sec\alpha$	$\pm\csc\alpha$

575.

	0°.	30°.	45°.	60°.	90°.	120°.	135°.	150°.	180°.
sin	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
cos	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{3}$	-1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
ctn	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	∞
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1
csc	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

576. $\sin\frac{1}{2}a = \sqrt{\frac{1}{2}(1-\cos a)}$.

577. $\cos\frac{1}{2}a = \sqrt{\frac{1}{2}(1+\cos a)}$.

578. $\tan\frac{1}{2}a = \sqrt{\dfrac{1-\cos a}{1+\cos a}} = \dfrac{1-\cos a}{\sin a} = \dfrac{\sin a}{1+\cos a}$.

579. $\sin 2a = 2\sin a\cos a$.

580. $\sin 3a = 3\sin a - 4\sin^3 a$.

581. $\sin 4a = 8\cos^3 a\cdot\sin a - 4\cos a\sin a$.

582. $\sin 5\alpha = 5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha.$

583. $\sin 6\alpha = 32\cos^5\alpha\sin\alpha - 32\cos^3\alpha\sin\alpha + 6\cos\alpha\sin\alpha.$

584. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1.$

585. $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha.$

586. $\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1.$

587. $\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha.$

588. $\cos 6\alpha = 32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1.$

589. $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}.$

590. $\operatorname{ctn} 2\alpha = \dfrac{\operatorname{ctn}^2\alpha - 1}{2\operatorname{ctn}\alpha}.$

591. $\sin(\alpha \pm \beta) = \sin\alpha\cdot\cos\beta \pm \cos\alpha\cdot\sin\beta.$

592. $\cos(\alpha \pm \beta) = \cos\alpha\cdot\cos\beta \mp \sin\alpha\cdot\sin\beta.$

593. $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\cdot\tan\beta}.$

594. $\operatorname{ctn}(\alpha \pm \beta) = \dfrac{\operatorname{ctn}\alpha\cdot\operatorname{ctn}\beta \mp 1}{\operatorname{ctn}\alpha \pm \operatorname{ctn}\alpha}.$

595. $\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha \pm \beta)\cdot\cos\frac{1}{2}(\alpha \mp \beta).$

596. $\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha + \beta)\cdot\cos\frac{1}{2}(\alpha - \beta).$

597. $\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha + \beta)\cdot\sin\frac{1}{2}(\alpha - \beta).$

598. $\tan\alpha \pm \tan\beta = \dfrac{\sin(\alpha \pm \beta)}{\cos\alpha\cdot\cos\beta}.$

599. $\operatorname{ctn}\alpha \pm \operatorname{ctn}\beta = \pm\dfrac{\sin(\alpha \pm \beta)}{\sin\alpha\cdot\sin\beta}.$

600. $\frac{\sin a \pm \sin \beta}{\cos a + \cos \beta} = \tan \frac{1}{2}(a \pm \beta).$

601. $\frac{\sin a \pm \sin \beta}{\cos a - \cos \beta} = -\operatorname{ctn} \frac{1}{2}(a \mp \beta).$

602. $\frac{\sin a + \sin \beta}{\sin a - \sin \beta} = \frac{\tan \frac{1}{2}(a + \beta)}{\tan \frac{1}{2}(a - \beta)}.$

603. $\sin^2 a - \sin^2 \beta = \sin(a + \beta) \cdot \sin(a - \beta).$

604. $\cos^2 a - \cos^2 \beta = -\sin(a + \beta) \cdot \sin(a - \beta).$

605. $\cos^2 a - \sin^2 \beta = \cos(a + \beta) \cdot \cos(a - \beta).$

606. $\sin xi = \frac{1}{2} i(e^x - e^{-x}) = i \sinh x.$

607. $\cos xi = \frac{1}{2}(e^x + e^{-x}) = \cosh x.$

608. $\tan xi = \frac{i(e^x - e^{-x})}{e^x + e^{-x}} = i \tanh x.$

609. $e^{x+yi} = e^x \cos y + ie^x \sin y.$

610. $a^{x+yi} = a^x \cos(y \cdot \log a) + ia^x \sin(y \cdot \log a).$

611. $(\cos \theta \pm i \cdot \sin \theta)^n = \cos n\theta \pm i \cdot \sin n\theta.$

612. $\sin x = -\frac{1}{2} i(e^{xi} - e^{-xi}).$

613. $\cos x = \frac{1}{2}(e^{xi} + e^{-xi}).$

614. $\tan x = -i\frac{e^{2xi} - 1}{e^{2xi} + 1}.$

615. $\sin(x \pm yi) = \sin x \cos yi \pm \cos x \sin yi$

$= \sin x \cosh y \pm i \cos x \sinh y.$

616. $\cos(x \pm yi) = \cos x \cos yi \mp \sin x \sin yi$

$= \cos x \cosh y \mp i \sin x \sinh y.$

In any plane triangle,

617. $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

618. $$a^2 = b^2 + c^2 - 2\,bc \cos A.$$

619. $$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} = \frac{\operatorname{ctn} \frac{1}{2} C}{\tan \frac{1}{2}(A-B)}.$$

620. $$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}, \text{ where } 2\,s = a + b + c.$$

621. $$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}.$$

622. $$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

623. $$\text{Area} = \tfrac{1}{2}\,bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}.$$

In any spherical triangle,

624. $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

625. $$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

626. $$-\cos A = \cos B \cos C - \sin B \sin C \cos a.$$

627. $$\sin a \operatorname{ctn} b = \sin C \operatorname{ctn} B + \cos a \cos C.$$

628. $$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \cdot \sin (s-a)}{\sin b \cdot \sin c}}.$$

629. $$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \cdot \sin (s-c)}{\sin b \cdot \sin c}}.$$

630. $$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \cdot \sin (s-c)}{\sin s \cdot \sin (s-a)}}.$$

631. $$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos (S - B) \cdot \cos (S - C)}{\sin B \cdot \sin C}}.$$

632. $$\sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cdot \cos (S - A)}{\sin B \sin C}}.$$

633. $$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cdot \cos (S - A)}{\cos (S - B) \cdot \cos (S - C)}}.$$

$$2s = a + b + c. \qquad 2S = A + B + C.$$

634. $$\cos \tfrac{1}{2} (A + B) = \frac{\cos \frac{1}{2} (a + b)}{\cos \frac{1}{2} c} \sin \tfrac{1}{2} C.$$

635. $$\cos \tfrac{1}{2} (A - B) = \frac{\sin \frac{1}{2} (a + b)}{\sin \frac{1}{2} c} \sin \tfrac{1}{2} C.$$

636. $$\sin \tfrac{1}{2} (A + B) = \frac{\cos \frac{1}{2} (a - b)}{\cos \frac{1}{2} c} \cos \tfrac{1}{2} C.$$

637. $$\sin \tfrac{1}{2} (A - B) = \frac{\sin \frac{1}{2} (a - b)}{\sin \frac{1}{2} c} \cos \tfrac{1}{2} C.$$

638. $$\tan \tfrac{1}{2} (A + B) = \frac{\cos \frac{1}{2} (a - b)}{\cos \frac{1}{2} (a + b)} \operatorname{ctn} \tfrac{1}{2} C.$$

639. $$\tan \tfrac{1}{2} (A - B) = \frac{\sin \frac{1}{2} (a - b)}{\sin \frac{1}{2} (a + b)} \operatorname{ctn} \tfrac{1}{2} C.$$

640. $$\tan \tfrac{1}{2} (a + b) = \frac{\cos \frac{1}{2} (A - B)}{\cos \frac{1}{2} (A + B)} \tan \tfrac{1}{2} c.$$

641. $$\tan \tfrac{1}{2} (a - b) = \frac{\sin \frac{1}{2} (A - B)}{\sin \frac{1}{2} (A + B)} \tan \tfrac{1}{2} c.$$

642. $$\frac{\cos \frac{1}{2} (a + b)}{\cos \frac{1}{2} (a - b)} = \frac{\operatorname{ctn} \frac{1}{2} C}{\tan \frac{1}{2} (A + B)}.$$

In interpreting equations which involve logarithmic and anti-trigonometric functions, it is necessary to remember that these functions are multiple valued. To save space the formulas on this page and the next are printed in contracted form.

643. $$\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \sec^{-1}\frac{1}{\sqrt{1-x^2}}$$
$$= \csc^{-1}\frac{1}{x} = 2\sin^{-1}[\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-x^2}]^{\frac{1}{2}}$$
$$= \tfrac{1}{2}\sin^{-1}(2x\sqrt{1-x^2}) = 2\tan^{-1}\left[\frac{1-\sqrt{1-x^2}}{x}\right]$$
$$= \tfrac{1}{2}\tan^{-1}\left[\frac{2x\sqrt{1-x^2}}{1-2x^2}\right] = \tfrac{1}{2}\pi - \cos^{-1}x$$
$$= \tfrac{1}{2}\pi - \sin^{-1}\sqrt{1-x^2} = -\sin^{-1}(-x)$$
$$= \operatorname{ctn}^{-1}\frac{\sqrt{1-x^2}}{x} = (2n+\tfrac{1}{2})\pi - i\log(x+\sqrt{x^2-1})$$
$$= \tfrac{1}{4}\pi + \tfrac{1}{2}\sin^{-1}(2x^2-1) = \tfrac{1}{2}\cos^{-1}(1-2x^2).$$

644. $$\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\frac{1}{x}$$
$$= \tfrac{1}{2}\pi - \sin^{-1}x = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$$
$$= \tfrac{1}{2}\cos^{-1}(2x^2-1)$$
$$= 2\tan^{-1}\sqrt{\frac{1-x}{1+x}} = \tfrac{1}{2}\tan^{-1}\left[\frac{2x\sqrt{1-x^2}}{2x^2-1}\right]$$
$$= \csc^{-1}\frac{1}{\sqrt{1-x^2}} = \pi - \cos^{-1}(-x)$$
$$= \operatorname{ctn}^{-1}\frac{x}{\sqrt{1-x^2}}$$
$$= i\log(x+\sqrt{x^2-1}) = \pi - i\log(\sqrt{x^2-1}-x).$$

645. $$\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \tfrac{1}{2}\sin^{-1}\frac{2x}{1+x^2}$$
$$= \operatorname{ctn}^{-1}\frac{1}{x} = \tfrac{1}{2}\pi - \operatorname{ctn}^{-1}x = \sec^{-1}\sqrt{1+x^2}$$
$$= \tfrac{1}{2}\pi - \tan^{-1}\frac{1}{x}$$
$$= \csc^{-1}\frac{\sqrt{1+x^2}}{x} = \tfrac{1}{2}\cos^{-1}\left[\frac{1-x^2}{1+x^2}\right]$$
$$= 2\cos^{-1}\left[\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}\right]^{\frac{1}{2}} = 2\sin^{-1}\left[\frac{\sqrt{1+x^2}-1}{2\sqrt{1+x^2}}\right]^{\frac{1}{2}}$$
$$= \tfrac{1}{2}\tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$$
$$= -\tan^{-1}c + \tan^{-1}\left[\frac{x+c}{1-cx}\right] = -\tan^{-1}(-x)$$
$$= \tfrac{1}{2}i\log\frac{1-xi}{1+xi} = \tfrac{1}{2}i\log\frac{i+x}{i-x}$$
$$= -\tfrac{1}{2}i\log\frac{1+xi}{1-xi}.$$

646. $$\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}].$$

647. $$\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}[xy \mp \sqrt{(1-x^2)(1-y^2)}].$$

648. $$\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left[\frac{x \pm y}{1 \mp xy}\right].$$

649. $$\sin^{-1}x \pm \cos^{-1}y = \sin^{-1}[xy \pm \sqrt{(1-x^2)(1-y^2)}]$$
$$= \cos^{-1}[y\sqrt{1-x^2} \mp x\sqrt{1-y^2}].$$

650. $$\tan^{-1}x \pm \operatorname{ctn}^{-1}y = \tan^{-1}\left[\frac{xy \pm 1}{y \mp x}\right] = \operatorname{ctn}^{-1}\left[\frac{y \mp x}{xy \pm 1}\right].$$

651. $$\log(x+yi) = \tfrac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x).$$

B. — Hyperbolic Functions.

652. $\sinh x = \frac{1}{2}(e^x - e^{-x}) = -\sinh(-x) = -i\sin(ix)$
$= (\operatorname{csch} x)^{-1} = 2\tanh\frac{1}{2}x \div (1 - \tanh^2\frac{1}{2}x).$

653. $\cosh x = \frac{1}{2}(e^x + e^{-x}) = \cosh(-x) = \cos(ix) = (\operatorname{sech} x)^{-1}$
$= (1 + \tanh^2\frac{1}{2}x) \div (1 - \tanh^2\frac{1}{2}x).$

654. $\tanh x = (e^x - e^{-x}) \div (e^x + e^{-x}) = -\tanh(-x)$
$= -i\tan(ix) = (\operatorname{ctnh} x)^{-1} = \sinh x \div \cosh x.$

655. $\cosh xi = \cos x.$

656. $\sinh xi = i\sin x.$

657. $\cosh^2 x - \sinh^2 x = 1.$

658. $1 - \tanh^2 x = \operatorname{sech}^2 x.$

659. $1 - \operatorname{ctnh}^2 x = -\operatorname{csch}^2 x.$

660. $\sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y.$

661. $\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y.$

662. $\tanh(x \pm y) = (\tanh x \pm \tanh y) \div (1 \pm \tanh x \cdot \tanh y).$

663. $\sinh(2x) = 2\sinh x\cosh x.$

664. $\cosh(2x) = \cosh^2 x + \sinh^2 y = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x.$

665. $\tanh(2x) = 2\tanh x \div (1 + \tanh^2 x).$

666. $\sinh(\frac{1}{2}x) = \sqrt{\frac{1}{2}(\cosh x - 1)}.$

667. $\cosh(\frac{1}{2}x) = \sqrt{\frac{1}{2}(\cosh x + 1)}.$

668. $\tanh(\frac{1}{2}x) = (\cosh x - 1) \div \sinh x = \sinh x \div (\cosh x + 1).$

669. $\sinh x + \sinh y = 2\sinh\frac{1}{2}(x + y) \cdot \cosh\frac{1}{2}(x - y).$

670. $\sinh x - \sinh y = 2\cosh\frac{1}{2}(x + y) \cdot \sinh\frac{1}{2}(x - y).$

671. $\cosh x + \cosh y = 2 \cosh \tfrac{1}{2}(x+y) \cdot \cosh \tfrac{1}{2}(x-y).$

672. $\cosh x - \cosh y = 2 \sinh \tfrac{1}{2}(x+y) \cdot \sinh \tfrac{1}{2}(x-y).$

673. $d \sinh x = \cosh x \cdot dx.$

674. $d \cosh x = \sinh x \cdot dx.$

675. $d \tanh x = \operatorname{sech}^2 x \cdot dx.$

676. $d \operatorname{ctnh} x = -\operatorname{csch}^2 x \cdot dx.$

677. $d \operatorname{sech} x = -\operatorname{sech} x \cdot \tanh x \cdot dx.$

678. $d \operatorname{csch} x = -\operatorname{csch} x \cdot \operatorname{ctnh} x \cdot dx.$

679. $$\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right) = \int \frac{dx}{\sqrt{x^2+1}}$$
$$= \cosh^{-1} \sqrt{x^2+1}.$$

680. $$\cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right) = \int \frac{dx}{\sqrt{x^2-1}}$$
$$= \sinh^{-1} \sqrt{x^2-1}.$$

681. $$\tanh^{-1} x = \tfrac{1}{2} \log(1+x) - \tfrac{1}{2} \log(1-x) = \int \frac{dx}{1-x^2}.$$

682. $$\operatorname{ctnh}^{-1} x = \tfrac{1}{2} \log(1+x) - \tfrac{1}{2} \log(x-1) = \int \frac{dx}{1-x^2}.$$

683. $$\operatorname{sech}^{-1} x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) = -\int \frac{dx}{x\sqrt{1-x^2}}.$$

684. $$\operatorname{csch}^{-1} x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right) = -\int \frac{dx}{x\sqrt{x^2+1}}.$$

685. $$d \sinh^{-1} x = \frac{dx}{\sqrt{1+x^2}}.$$

686. $$d \cosh^{-1} x = \frac{dx}{\sqrt{x^2-1}}.$$

687. $d \tanh^{-1} x = \frac{dx}{1-x^2}.$

688. $d \operatorname{ctnh}^{-1} x = -\frac{dx}{x^2-1}.$

689. $d \operatorname{sech}^{-1} x = -\frac{dx}{x\sqrt{1-x^2}}.$

690. $d \operatorname{csch}^{-1} x = -\frac{dx}{x\sqrt{x^2+1}}.$

If m is an integer,

691. $\sinh(m\pi i) = 0.$

692. $\cosh(m\pi i) = \cos m\pi = (-1)^m.$

693. $\tanh(m\pi i) = 0.$

694. $\sinh(x + m\pi i) = (-1)^m \sinh x.$

695. $\cosh(x + m\pi i) = (-1)^m \cosh(x).$

696. $\sinh(2m+1)\frac{1}{2}\pi i = i \sin(2m+1)\frac{1}{2}\pi = \pm i.$

697. $\cosh(2m+1)\frac{1}{2}\pi i = 0.$

698. $\sinh\left(\frac{\pi i}{2} \pm x\right) = i \cosh x.$

799. $\cosh\left(\frac{\pi i}{2} \pm x\right) = \pm i \sinh x.$

700. $\sinh u = \tan \operatorname{gd} u.$

701. $\cosh u = \sec \operatorname{gd} u.$

702. $\tanh u = \sin \operatorname{gd} u.$

703. $\tanh \frac{1}{2} u = \tan \frac{1}{2} \operatorname{gd} u.$

704. $u = \log \tan(\frac{1}{4}\pi + \frac{1}{2} \operatorname{gd} u).$

C. — ELLIPTIC FUNCTIONS.

If $u \equiv F(\phi, k) \equiv \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \equiv \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$, where $k < 1$, and $x \equiv \sin\phi$, ϕ is called the *amplitude* of u and is written $\text{am}\,(u, \text{mod}\,k)$, or, more simply, $\text{am}\,u$; $x \equiv \sin\phi \equiv \text{sn}\,u$,

$$\sqrt{1-x^2} \equiv \cos\phi \equiv \text{cn}\,u, \quad \sqrt{1-k^2x^2} \equiv \Delta\phi \equiv \Delta\text{n}\,u \equiv \text{dn}\,u,$$

$$K \equiv F(\tfrac{1}{2}\pi, k), \qquad K' \equiv F(\tfrac{1}{2}\pi, k').$$

Hence, $\text{am}\,(0) = 0,\quad \text{sn}\,(0) = 0,\quad \text{cn}\,(0) = 1,\quad \text{dn}\,(0) = 1,$

$$\text{am}\,(-u) = -\,\text{am}\,u, \qquad \text{sn}\,(-u) = -\,\text{sn}\,u,$$

$$\text{cn}\,(-u) = \text{cn}\,u, \qquad \text{dn}\,(-u) = \text{dn}\,u.$$

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705. $\text{sn}^2 u + \text{cn}^2 u = 1.$

706. $\text{dn}^2 u + k^2\,\text{sn}^2 u = 1.$

707. $\text{dn}^2 u - k^2\,\text{cn}^2 u = 1 - k^2 = k'^2.$

708. $$\text{sn}\,2u = \frac{2\,\text{sn}\,u \cdot \text{cn}\,u \cdot \text{dn}\,u}{1 - k^2\,\text{sn}^4 u}.$$

709. $$\text{cn}\,2u = \frac{\text{cn}^2 u - \text{sn}^2 u \cdot \text{dn}^2 u}{1 - k^2\,\text{sn}^4 u} = \frac{1 - 2\,\text{sn}^2 u + k^2\,\text{sn}^4 u}{1 - k^2\,\text{sn}^4 u}$$
$$= 1 - \frac{2\,\text{sn}^2 u \cdot \text{dn}^2 u}{1 - k^2\,\text{sn}^4 u} = \frac{2\,\text{cn}^2 u}{1 - k^2\,\text{sn}^4 u} - 1.$$

710. $$\text{dn}\,2u = \frac{\text{dn}^2 u - k^2\,\text{sn}^2 u \cdot \text{cn}^2 u}{1 - k^2\,\text{sn}^4 u} = \frac{1 - 2k^2\,\text{sn}^2 u + k^2\,\text{sn}^4 u}{1 - k^2\,\text{sn}^4 u}$$
$$= 1 - \frac{2k^2\,\text{sn}^2 u \cdot \text{cn}^2 u}{1 - k^2\,\text{sn}^4 u} = \frac{2\,\text{dn}^2 u}{1 - k^2\,\text{sn}^4 u} - 1.$$

711. $$\text{sn}^2\left(\frac{u}{2}\right) = \frac{1 - \text{cn}\,u}{1 + \text{dn}\,u} = \frac{1 - \text{dn}\,u}{k^2(1 + \text{cn}\,u)} = \frac{\text{dn}\,u - \text{cn}\,u}{k'^2 + \text{dn}\,u - k^2\,\text{cn}\,u}.$$

712. $$\text{cn}^2\left(\frac{u}{2}\right) = \frac{\text{dn}\,u + \text{cn}\,u}{1 + \text{dn}\,u} = \frac{k^2\,\text{cn}\,u - k'^2 + \text{dn}\,u}{k^2(1 + \text{cn}\,u)}$$
$$= \frac{k'^2(1 + \text{cn}\,u)}{k'^2 + \text{dn}\,u - k^2\,\text{cn}\,u}.$$

713. $$\mathrm{dn}^2\left(\frac{u}{2}\right) = \frac{k'^2 + \mathrm{dn}\, u + k^2\, \mathrm{cn}\, u}{1 + \mathrm{dn}\, u} = \frac{k^2\,(\mathrm{cn}\, u + \mathrm{dn}\, u)}{k^2\,(1 + \mathrm{cn}\, u)}$$
$$= \frac{k'^2(1 + \mathrm{dn}\, u)}{k'^2 + \mathrm{dn}\, u - k^2\, \mathrm{cn}\, u}.$$

If, moreover, $v = \int_0^y \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}$,

714. $\mathrm{sn}^2 u - \mathrm{sn}^2 v = \mathrm{cn}^2 v - \mathrm{cn}^2 u.$

715. $$\mathrm{sn}\,(u \pm v) = \frac{\mathrm{sn}\, u \cdot \mathrm{cn}\, v \cdot \mathrm{dn}\, v \pm \mathrm{cn}\, u \cdot \mathrm{sn}\, v \cdot \mathrm{dn}\, u}{1 - k^2\, \mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}.$$

716. $$\mathrm{cn}\,(u \pm v) = \frac{\mathrm{cn}\, u \cdot \mathrm{cn}\, v \mp \mathrm{sn}\, u \cdot \mathrm{sn}\, v \cdot \mathrm{dn}\, u \cdot \mathrm{dn}\, v}{1 - k^2\, \mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}$$
$$= \mathrm{cn}\, u \cdot \mathrm{cn}\, v \mp \mathrm{sn}\, u \cdot \mathrm{sn}\, v \cdot \mathrm{dn}\,(u \pm v).$$

717. $$\mathrm{dn}\,(u \pm v) = \frac{\mathrm{dn}\, u \cdot \mathrm{dn}\, v \mp k^2\, \mathrm{sn}\, u \cdot \mathrm{sn}\, v \cdot \mathrm{cn}\, u \cdot \mathrm{cn}\, v}{1 - k^2\, \mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}$$
$$= \mathrm{dn}\, u \cdot \mathrm{dn}\, v \mp k^2\, \mathrm{sn}\, u \cdot \mathrm{sn}\, v \cdot \mathrm{cn}\,(u \pm v).$$

718. $$\mathrm{tn}\,(u \pm v) = \frac{\mathrm{tn}\, u \cdot \mathrm{dn}\, v \pm \mathrm{tn}\, v \cdot \mathrm{dn}\, u}{1 \mp \mathrm{tn}\, u \cdot \mathrm{tn}\, v \cdot \mathrm{dn}\, u \cdot \mathrm{dn}\, v}.$$

719. $$\mathrm{sn}\,(u + v) + \mathrm{sn}\,(u - v) = \frac{2\, \mathrm{sn}\, u \cdot \mathrm{cn}\, v \cdot \mathrm{dn}\, v}{1 - k^2\, \mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}.$$

720. $$\mathrm{sn}\,(u + v) - \mathrm{sn}\,(u - v) = \frac{2\, \mathrm{sn}\, v \cdot \mathrm{cn}\, u \cdot \mathrm{dn}\, u}{1 - k^2\, \mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}.$$

721. $$\mathrm{cn}\,(u + v) + \mathrm{cn}\,(u - v) = \frac{2\, \mathrm{cn}\, u \cdot \mathrm{cn}\, v}{1 - k^2\, \mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}.$$

722. $$\mathrm{cn}\,(u + v) - \mathrm{cn}\,(u - v) = -\frac{2\, \mathrm{sn}\, u \cdot \mathrm{sn}\, v \cdot \mathrm{dn}\, u \cdot \mathrm{dn}\, v}{1 - k^2\, \mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}.$$

723. $$\mathrm{dn}\,(u + v) + \mathrm{dn}\,(u - v) = \frac{2\, \mathrm{dn}\, u \cdot \mathrm{dn}\, v}{1 - k^2\, \mathrm{sn}^2 u \cdot \mathrm{sn}^2 v}.$$

724. $$\mathrm{dn}\,(u+v)-\mathrm{dn}\,(u-v)=-\frac{2\,k^2\,\mathrm{sn}\,u\cdot\mathrm{sn}\,v\cdot\mathrm{cn}\,u\cdot\mathrm{cn}\,v}{1-k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}.$$

725. $$\mathrm{sn}\,(u+v)\cdot\mathrm{sn}\,(u-v)=\frac{\mathrm{sn}^2 u-\mathrm{sn}^2 v}{1-k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}$$

$$=\frac{\mathrm{cn}^2 v+\mathrm{sn}^2 u\cdot\mathrm{dn}^2 v}{1-k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}-1=\frac{1}{k^2}\left[\frac{\mathrm{dn}^2 v+k^2\,\mathrm{sn}^2 u\cdot\mathrm{cn}^2 v}{1-k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}-1\right].$$

726. $$\mathrm{cn}\,(u+v)\cdot\mathrm{cn}\,(u-v)=\frac{\mathrm{cn}^2 u-\mathrm{sn}^2 v+k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}{1-k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}$$

$$=\frac{\mathrm{cn}^2 u+\mathrm{cn}^2 v}{1-k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}-1=1-\frac{\mathrm{sn}^2 u\cdot\mathrm{dn}^2 v+\mathrm{sn}^2 v\cdot\mathrm{dn}^2 u}{1-k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}.$$

727. $$\mathrm{dn}\,(u+v)\cdot\mathrm{dn}\,(u-v)$$

$$=\frac{1-k^2\,\mathrm{sn}^2 u-k^2\,\mathrm{sn}^2 v+k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}{1-k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}$$

$$=\frac{\mathrm{dn}^2 u+\mathrm{dn}^2 v}{1-k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}-1.$$

728. $$\mathrm{sn}\,(u\pm v)\,\mathrm{cn}\,(u\mp v)=\frac{\mathrm{sn}\,u\cdot\mathrm{cn}\,u\cdot\mathrm{dn}\,v\pm\mathrm{sn}\,v\cdot\mathrm{cn}\,v\cdot\mathrm{dn}\,u}{1-k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}.$$

729. $$\mathrm{sn}\,(u\pm v)\,\mathrm{dn}\,(u\mp v)=\frac{\mathrm{sn}\,u\cdot\mathrm{dn}\,u\cdot\mathrm{cn}\,v\pm\mathrm{sn}\,v\cdot\mathrm{dn}\,v\cdot\mathrm{cn}\,u}{1-k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}.$$

730. $$\mathrm{cn}\,(u\pm v)\,\mathrm{dn}\,(u\mp v)=\frac{\mathrm{cn}\,u\cdot\mathrm{dn}\,u\cdot\mathrm{cn}\,v\cdot\mathrm{dn}\,v\mp k'^2\,\mathrm{sn}\,u\cdot\mathrm{sn}\,v}{1-k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}.$$

731. $$[1\pm\mathrm{sn}\,(u+v)][1\pm\mathrm{sn}\,(u-v)]=\frac{(\mathrm{cn}\,v\pm\mathrm{sn}\,u\cdot\mathrm{dn}\,v)^2}{1-k^2\,\mathrm{sn}^2 u\cdot\mathrm{sn}^2 v}.$$

732. $\mathrm{sn}\,(ui,\,k)=i\,\mathrm{sn}\,(u,\,k')/\mathrm{cn}\,(u,\,k')$.

733. $\mathrm{cn}\,(ui,\,k)=1/\mathrm{cn}\,(u,\,k')$.

734. $\mathrm{dn}\,(ui,\,k)=\mathrm{dn}\,(u,\,k')/\mathrm{cn}\,(u,\,k')$.

735. $\dfrac{d\ \mathrm{am}\ u}{du} = \mathrm{dn}\ u.$

736. $\dfrac{d\ \mathrm{sn}\ u}{du} = \mathrm{cn}\ u \cdot \mathrm{dn}\ u.$

737. $\dfrac{d\ \mathrm{cn}\ u}{du} = -\ \mathrm{sn}\ u \cdot \mathrm{dn}\ u.$

738. $\dfrac{d\ \mathrm{dn}\ u}{du} = -\ k^2\ \mathrm{sn}\ u \cdot \mathrm{cn}\ u.$

739. $\dfrac{d^2\ \mathrm{sn}\ u}{du^2} = 2\ k^2\ \mathrm{sn}^3\ u - (1 + k^2)\ \mathrm{sn}\ u.$

740. $\dfrac{d^2\ \mathrm{cn}\ u}{du^2} = (2\ k^2 - 1)\ \mathrm{cn}\ u - 2\ k^2\ \mathrm{cn}^3\ u.$

741. $\dfrac{d^2\ \mathrm{dn}\ u}{du^2} = (2 - k^2)\ \mathrm{dn}\ u - 2\ \mathrm{dn}^3\ u.$

742. $\mathrm{sn}\,(u \pm K) = \pm\ \mathrm{cn}\ u / \mathrm{dn}\ u,$ $\mathrm{sn}\,(u \pm 2\ K) = -\ \mathrm{sn}\ u,$
$\mathrm{sn}\,(u \pm 3\ K) = \mp\ \mathrm{cn}\ u / \mathrm{dn}\ u,$ $\mathrm{sn}\,(u \pm 4\ K) = \mathrm{sn}\ u,$
$\mathrm{sn}\,(u + K'i) = 1 / k\ \mathrm{sn}\ u,$ and, if m and n are integers,
$\mathrm{sn}\,(u + 2\ mK + 2\ nK'i) = (-1)^m\ \mathrm{sn}\ u.$

743. $\mathrm{cn}\,(u \pm K) = \mp\ k'\ \mathrm{sn}\ u / \mathrm{dn}\ u,$ $\mathrm{cn}\,(u \pm 2\ K) = -\ \mathrm{cn}\ u,$
$\mathrm{cn}\,(u \pm 3\ K) = \pm\ k'\ \mathrm{sn}\ u / \mathrm{dn}\ u,$ $\mathrm{cn}\,(u \pm 4\ K) = \mathrm{cn}\ u,$
$\mathrm{cn}\,(u + K'i) = -\ i\ \mathrm{dn}\ u / k\ \mathrm{sn}\ u,$ and, if m and n are integers,
$\mathrm{cn}\,(u + 2\ mK + 2\ nK'i) = (-1)^{m+n}\ \mathrm{cn}\ u.$

744. $\mathrm{dn}\ (u \pm K) = k' / \mathrm{dn}\ u,$ $\mathrm{dn}\,(u \pm 2\ K) = \mathrm{dn}\ u,$
$\mathrm{dn}\,(u + K'i) = -\ i\ \mathrm{cn}\ u / \mathrm{sn}\ u,$ and, if m and n are integers,
$\mathrm{dn}\,(u + 2\ mK + 2\ nK'i) = (-1)^n\ \mathrm{dn}\ u.$

D. — Series and Products.

[The expression in brackets attached to an infinite series shows values of the variable which lie within the interval of convergence. If a series is convergent for all finite values of x, the expression $[x^2 < \infty]$ is used.]

745. $$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \frac{n!\,a^{n-k}b^k}{(n-k)!\,k!} + \cdots. \quad [b^2 < a^2.]$$

746. $$(a-bx)^{-1} = \frac{1}{a}\left[1 + \frac{bx}{a} + \frac{b^2x^2}{a^2} + \frac{b^3x^3}{a^3} + \cdots\right]\cdot \quad [b^2x^2 < a^2.]$$

747. $$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots + \frac{(\pm 1)^k\, n!\, x^k}{(n-k)!\,k!} + \cdots. \quad [x^2 < 1.]$$

748. $$(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)}{2!}x^2 \mp \frac{n(n+1)(n+2)x^3}{3!} + \cdots (\mp)^k \frac{(n+k-1)!\,x^k}{(n-1)!\,k!} + \cdots. \quad [x^2 < 1.]$$

749. $$(1 \pm x)^{\frac{1}{2}} = 1 \pm \tfrac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 \pm \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 \pm \cdots. \quad [x^2 < 1.]$$

750. $$(1 \pm x)^{-\frac{1}{2}} = 1 \mp \tfrac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 \mp \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 \mp \cdots. \quad [x^2 < 1.]$$

751. $$(1 \pm x)^{\frac{1}{3}} = 1 \pm \tfrac{1}{3}x - \frac{1\cdot 2}{3\cdot 6}x^2 \pm \frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}x^3 - \frac{1\cdot 2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}x^4 \pm \cdots. \quad [x^2 < 1.]$$

752. $$(1 \pm x)^{-\frac{1}{3}} = 1 \mp \tfrac{1}{3}x + \frac{1 \cdot 4}{3 \cdot 6}x^2 \mp \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9}x^3 + \frac{1 \cdot 4 \cdot 7 \cdot 10}{3 \cdot 6 \cdot 9 \cdot 12}x^4 \mp \cdots. \qquad [x^2 < 1.]$$

753. $$(1 \pm x^2)^{\frac{1}{2}} = 1 \pm \tfrac{1}{2}x^2 - \frac{x^4}{2 \cdot 4} \pm \frac{1 \cdot 3 x^6}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5 x^8}{2 \cdot 4 \cdot 6 \cdot 8} \pm \cdots. \qquad [x^2 < 1.]$$

754. $$(1 \pm x^2)^{-\frac{1}{2}} = 1 \mp \tfrac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 \mp \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \cdots. \qquad [x^2 < 1.]$$

755. $$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \cdots. \qquad [x^2 < 1.]$$

756. $$(1 \pm x)^{\frac{3}{2}} = 1 \pm \tfrac{3}{2}x + \frac{3 \cdot 1}{2 \cdot 4}x^2 \mp \frac{3 \cdot 1 \cdot 1}{2 \cdot 4 \cdot 6}x^3 + \frac{3 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6 \cdot 8}x^4 \mp \frac{3 \cdot 1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}x^5 + \cdots. \qquad [x^2 < 1.]$$

757. $$(1 \pm x)^{-\frac{3}{2}} = 1 \mp \tfrac{3}{2}x + \frac{3 \cdot 5}{2 \cdot 4}x^2 \mp \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6}x^3 + \cdots. \qquad [x^2 < 1.]$$

758. $$(1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + \cdots. \qquad [x^2 < 1.]$$

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759. $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \qquad [x^2 < \infty.]$$

760. $$a^x = 1 + x \log a + \frac{(x \log a)^2}{2!} + \frac{(x \log a)^3}{3!} + \cdots. \qquad [x^2 < \infty.]$$

761. $$\tfrac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots. \qquad [x^2 < \infty.]$$

762. $$\tfrac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots. \qquad [x^2 < \infty.]$$

763. $$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots. \qquad [x^2 < \infty.]$$

A series of numbers, B_1, B_2, $B_3 \cdots$, of odd and even orders, which appear in the developments of many functions, may be computed by means of the equations,

$$B_{2n} - \frac{2n(2n-1)}{2!} B_{2n-2}$$

$$+ \frac{2n(2n-1)(2n-2)(2n-3)}{4!} B_{2n-4} - \cdots (-1)^n \quad = 0.$$

$$\frac{2^{2n}(2^{2n}-1)}{2n} B_{2n-1} = (2n-1) B_{2n-2}$$

$$- \frac{(2n-1)(2n-2)(2n-3)}{3!} B_{2n-4} + \cdots (-1)^{n-1} = 0.$$

Whence $B_1 = \frac{1}{6}$, $B_2 = 1$, $B_3 = \frac{1}{30}$, $B_4 = 5$, $B_5 = \frac{1}{42}$, $B_6 = 61$, $B_7 = \frac{1}{30}$, $B_8 = 1385$, $B_9 = \frac{5}{66}$, $B_{10} = 50521$, $B_{11} = \frac{691}{2730}$, $B_{12} = 2702765$, $B_{13} = \frac{7}{6}$, etc. The B's of odd orders are called Bernoulli's Numbers; those of even orders, Euler's Numbers. What are here denoted by B_{2n-1} and B_{2n} are sometimes represented by B_n and E_n, respectively,

$$\frac{B_{2n-1}}{(2n)!} = \frac{2}{(2^{2n}-1)\pi^{2n}} \left[1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \cdots \right],$$

$$\frac{B_{2n}}{(2n)!} = \frac{2^{2n+2}}{\pi^{2n+1}} \left[1 - \frac{1}{3^{2n}} + \frac{1}{5^{2n}} - \frac{1}{7^{2n}} + \cdots \right].$$

764. $$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B_1 x^2}{2!} - \frac{B_3 x^4}{4!} + \frac{B_5 x^6}{6!} - \frac{B_7 x^8}{8!} + \cdots.$$ $[x < 2\pi.]$

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765. $\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots.$ $[2 > x > 0.]$

766. $$\log x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots.$$ $[x > \frac{1}{2}.]$

767. $$\log x = 2\left[\frac{x-1}{x+1} + \tfrac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \tfrac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right]. \qquad [x > 0.]$$

768. $$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots. \qquad [x^2 < 1.]$$

769. $$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \cdots\right]. \qquad [x^2 < 1.]$$

770. $$\log\left(\frac{x+1}{x-1}\right) = 2\left[\frac{1}{x} + \tfrac{1}{3}\left(\frac{1}{x}\right)^3 + \tfrac{1}{5}\left(\frac{1}{x}\right)^5 + \cdots\right]. \qquad [x^2 > 1.]$$

771. $$\log(x + \sqrt{1+x^2}) = x - \frac{1\,x^3}{6} + \frac{1\cdot 3\,x^5}{2\cdot 4\cdot 5} - \frac{1\cdot 3\cdot 5\,x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots. \qquad [x^2 < 1.]$$

Series for denary and other logarithms can be obtained from the foregoing developments by aid of the equations,

$$\log_a x = \log_e x \cdot \log_a e, \quad \log_e x = \log_a x \cdot \log_e a,$$
$$\log_e(-z) = (2n+1)\pi i + \log_e z.$$

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772. $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \qquad [x^2 < \infty.]$$

773. $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = 1 - \operatorname{versin} x. \qquad [x^2 < \infty.]$$

774. $$\tan x = x + \frac{x^3}{3} + \frac{2\,x^5}{15} + \frac{17\,x^7}{315} + \frac{62\,x^9}{2835}$$
$$+ \cdots + \frac{2^{2n}(2^{2n}-1)\,B_{2n-1}\,x^{2n-1}}{(2n)!} + \cdots. \qquad [x^2 < \tfrac{1}{4}\pi^2.]$$

775. $$\operatorname{ctn} x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2\,x^5}{945} - \frac{x^7}{4725}$$
$$- \cdots - \frac{B_{2n-1}(2x)^{2n}}{x\,(2n)!} - \cdots. \qquad [x^2 < \pi^2.]$$

776. $$\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots + \frac{B_{2n}x^{2n}}{(2n)!} + \cdots. \left[x^2 < \frac{\pi^2}{4}.\right]$$

777. $$\csc x = \frac{1}{x} + \frac{x}{3!} + \frac{7x^3}{3\cdot 5!} + \frac{31x^5}{3\cdot 7!}$$
$$+ \cdots + \frac{2(2^{2n+1}-1)}{(2n+2)!}B_{2n+1}x^{2n+1} + \cdots. \qquad [x^2 < \pi^2.]$$

778. $$\sin^{-1}x = x + \frac{x^3}{6} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}$$
$$+ \cdots = \tfrac{1}{2}\pi - \cos^{-1}x. \qquad [x^2 < 1.]$$

779. $$\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \tfrac{1}{2}\pi - \operatorname{ctn}^{-1}x. \qquad [x^2 < 1.]$$

780. $$\tan^{-1}x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots. \qquad [x^2 > 1.]$$

781. $$\sec^{-1}x = \frac{\pi}{2} - \frac{1}{x} - \frac{1}{6x^3} - \frac{1\cdot 3}{2\cdot 4\cdot 5x^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7x^7} - \cdots$$
$$= \tfrac{1}{2}\pi - \csc^{-1}x. \qquad [x^2 > 1.]$$

782. $$\log\sin x = \log x - \tfrac{1}{6}x^2 - \tfrac{1}{180}x^4 - \tfrac{1}{2835}x^6$$
$$- \cdots - \frac{2^{2n-1}B_{2n-1}x^{2n}}{n(2n)!} - \cdots. \qquad [x^2 < \pi^2.]$$

783. $$\log\cos x = -\tfrac{1}{2}x^2 - \tfrac{1}{12}x^4 - \tfrac{1}{45}x^6 - \tfrac{17}{2520}x^8$$
$$- \cdots - \frac{2^{2n-1}(2^{2n}-1)B_{2n-1}x^{2n}}{n(2n)!} - \cdots. \qquad [x^2 < \tfrac{1}{4}\pi^2.]$$

784. $$\log\tan x = \log x + \tfrac{1}{3}x^2 + \tfrac{7}{90}x^4 + \tfrac{62}{2835}x^6$$
$$+ \cdots + \frac{(2^{2n-1}-1)2^{2n}B_{2n-1}x^{2n}}{n(2n)!} + \cdots. \qquad [x^2 < \tfrac{1}{4}\pi^2.]$$

785. $$e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \frac{56x^7}{7} + \cdots.$$
$$[x^2 < \infty.]$$

786. $e^{\cos x} = e\left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \cdots\right).$ $[x^2 < \infty.]$

787. $e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \cdots.$ $[x^2 < \frac{1}{4}\pi^2.]$

788. $e^{\sin^{-1} x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \cdots.$ $[x^2 < 1.]$

789. $e^{\tan^{-1} x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7x^4}{24} - \cdots.$ $[x^2 < 1.]$

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790. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots.$ $[x^2 < \infty.]$

791. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots.$ $[x^2 < \infty.]$

792. $\tanh x = (2^2 - 1)2^2 B_1 \frac{x}{2!} - (2^4 - 1)2^4 B_3 \frac{x^3}{4!} + \cdots$

$$= \Sigma[(-1)^{n-1} 2^{2n}(2^{2n} - 1) B_{2n-1} x^{2n-1}/(2n)!].$$

$[x^2 < \frac{1}{4}\pi^2.]$

793. $\operatorname{ctnh} x = \frac{1}{x}(1 + \Sigma[(-1)^{n-1} 2^{2n} B_{2n-1} x^{2n}/(2n)!]).$

$[x^2 < \pi^2.]$

794. $\operatorname{sech} x = 1 + \Sigma[(-1)^n B_{2n} x^{2n}/(2n)!].$ $[x^2 < \frac{1}{4}\pi^2.]$

795. $\operatorname{csch} x = \frac{1}{x} + (2 - 1)2 B_1 \frac{x}{2!} - (2^3 - 1)2 B_3 \frac{x^3}{4!} + \cdots$

$$= \frac{1}{x}(1 + 2\Sigma[(-1)^n (2^{2n-1} - 1) B_{2n-1} x^{2n}/(2n)!]).$$

$[x^2 < \pi^2.]$

796. $\sinh^{-1} x = x - \frac{1}{6}x^3 + \frac{1 \cdot 3 \cdot x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5 \cdot x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots.$ $[x^2 < 1.]$

797. $$\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots. \qquad [x^2 < 1.]$$

798. $$\operatorname{ctnh}^{-1} x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots. \qquad [x^2 > 1.]$$

799. $$\operatorname{csch}^{-1} x = \frac{1}{x} - \frac{1}{2\cdot 3\cdot x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot x^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot x^7} + \cdots. \qquad [x^2 > 1.]$$

800. $$\int_0^x e^{-x^2}\,dx = x - \tfrac{1}{3}x^3 + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots. \qquad [x^2 < \infty.]$$

801. $$\int_0^x \cos(x^2)\,dx = x - \frac{x^5}{5\cdot 2!} + \frac{x^9}{9\cdot 4!} - \frac{x^{13}}{13\cdot 6!} + \cdots. \qquad [x^2 < \infty.]$$

802. $$\int_0^1 \frac{x^{a-1}\,dx}{1+x^b} = \frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \cdots.$$

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803. $$f(x+h) = f(x) + h\cdot f'(x+\theta h).$$

804. $$f(x+h) = f(x) + h\cdot f'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}\cdot f^n(x+\theta h).$$

805. $$f(x+h) = f(x) + h\cdot f'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{(n-1)!}\cdot(1-\theta)^{n-1}\cdot f^n(x+\theta h).$$

806. $$f(x+h, y+k) = f(x, y) + hf'_x(x+\theta h, y+\theta k) + kf'_y(x+\theta h, y+\theta k).$$

807. $$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f(x, y)}{\partial x} + k\frac{\partial f(x, y)}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f(x, y)}{\partial x^2} + 2hk\frac{\partial^2 f(x, y)}{\partial x\cdot\partial y} + k^2\frac{\partial^2 f(x, y)}{\partial y^2}\right)$$

$$+\frac{1}{3!}\left(h^3\frac{\partial^3 f(x,y)}{\partial x^3}+3\,h^2k\,\frac{\partial^3 f(x,y)}{\partial y\cdot\partial x^2}+3\,hk^2\,\frac{\partial^3 f(x,y)}{\partial x\cdot\partial y^2}\right.$$

$$\left.+\,k^3\frac{\partial f(x,y)}{\partial y^3}\right)+\cdots+R_n$$

$$=f(x,y)+(hD_x+kD_y)f(x,y)+\frac{1}{2!}(hD_x+kD_y)^2f(x,y)$$

$$+\cdots+\frac{1}{(n-1)!}(hD_x+kD_y)^{n-1}f(x,y)$$

$$+\frac{1}{n!}(hD_x+kD_y)^n f(x+\theta h,\,y+\theta k).$$

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808. $$1=\frac{4}{\pi}\left[\sin\frac{\pi x}{c}+\tfrac{1}{3}\sin\frac{3\,\pi x}{c}+\tfrac{1}{5}\sin\frac{5\,\pi x}{c}+\cdots\right].$$ $[0<x<c.]$

809. $$x=\frac{2\,c}{\pi}\left[\sin\frac{\pi x}{c}-\tfrac{1}{2}\sin\frac{2\,\pi x}{c}+\tfrac{1}{3}\sin\frac{3\,\pi x}{c}-\cdots\right].$$ $[-c<x<c.]$

810. $$x=\frac{c}{2}-\frac{4\,c}{\pi^2}\left[\cos\frac{\pi x}{c}+\frac{1}{3^2}\cos\frac{3\,\pi x}{c}+\frac{1}{5^2}\cos\frac{5\,\pi x}{c}+\cdots\right].$$ $[0<x<c.]$

811. $$x^2=\frac{2\,c^2}{\pi^3}\left[\left(\frac{\pi^2}{1}-\frac{4}{1}\right)\sin\frac{\pi x}{c}-\frac{\pi^2}{2}\sin\frac{2\,\pi x}{c}\right.$$

$$+\left(\frac{\pi^2}{3}-\frac{4}{3^3}\right)\sin\frac{3\,\pi x}{c}-\frac{\pi^2}{4}\sin\frac{4\,\pi x}{c}$$

$$\left.+\left(\frac{\pi^2}{5}-\frac{4}{5^3}\right)\sin\frac{5\,\pi x}{c}+\cdots\right].$$ $[0<x<c.]$

812. $$x^2=\frac{c^2}{3}-\frac{4\,c^2}{\pi^2}\left[\cos\frac{\pi x}{c}-\frac{1}{2^2}\cos\frac{2\,\pi x}{c}+\frac{1}{3^2}\cos\frac{3\,\pi x}{c}\right.$$

$$\left.-\frac{1}{4^2}\cos\frac{4\,\pi x}{c}+\cdots\right].$$ $[-c<x<c.]$

813. $$\log \sin \tfrac{1}{2} x = -\log 2 - \cos x - \tfrac{1}{2} \cos 2x - \tfrac{1}{3} \cos 3x - \cdots. \qquad [0 < x < \tfrac{1}{2}\pi.]$$

814. $$\log \cos \tfrac{1}{2} x = -\log 2 + \cos x - \tfrac{1}{2} \cos 2x + \tfrac{1}{3} \cos 3x - \cdots. \qquad [0 < x < \tfrac{1}{2}\pi.]$$

815. $$f(x) = \tfrac{1}{2} b_0 + b_1 \cos \frac{\pi x}{c} + b_2 \cos \frac{2\pi x}{c} + \cdots$$
$$+ a_1 \sin \frac{\pi x}{c} + a_2 \sin \frac{2\pi x}{c} + \cdots, \quad [-c < x < c.]$$

where $$b_m = \frac{1}{c} \int_{-c}^{+c} f(\alpha) \cos \frac{m\pi\alpha}{c}\, d\alpha,$$
$$a_m = \frac{1}{c} \int_{-c}^{+c} f(\alpha) \sin \frac{m\pi\alpha}{c}\, d\alpha.$$

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816. $$\sin\theta = \theta \left[1 - \left(\frac{\theta}{\pi}\right)^2\right] \left[1 - \left(\frac{\theta}{2\pi}\right)^2\right] \left[1 - \left(\frac{\theta}{3\pi}\right)^2\right] \cdots. \qquad [\theta^2 < \infty.]$$

817. $$\cos\theta = \left[1 - \left(\frac{2\theta}{\pi}\right)^2\right] \left[1 - \left(\frac{2\theta}{3\pi}\right)^2\right] \left[1 - \left(\frac{2\theta}{5\pi}\right)^2\right] \cdots. \qquad [\theta^2 < \infty.]$$

818. $$\frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2m)^2 (2m+2)}{1^2 \cdot 3^2 \cdot 5^2 \cdots (2m+1)^2} > \frac{\pi}{2}$$
$$> \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2m)^2 (2m+1)}{1^2 \cdot 3^2 \cdot 5^2 \cdots (2m+1)^2}.$$

819. $$J_n(x) = \frac{x^n}{2^n n!} \left\{ 1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4\,(2n+2)(2n+4)} \right.$$
$$\left. - \frac{x^6}{2 \cdot 4 \cdot 6\,(2n+2)(2n+4)(2n+6)} + \cdots \right\}.$$

E. — DERIVATIVES.

820. $\frac{d(au)}{dx} = \frac{a\,du}{dx}.$

821. $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$

822. $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$

823. $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$

824. $\frac{df(u)}{dx} = \frac{df(u)}{du}\cdot\frac{du}{dx}.$

825. $\frac{d^2f(u)}{dx^2} = \frac{df}{du}\cdot\frac{d^2u}{dx^2} + \frac{d^2f}{du^2}\cdot\frac{du^2}{dx^2}.$

826. $\frac{dx^n}{dx} = nx^{n-1}.$

827. $\frac{de^x}{dx} = e^x.$

828. $\frac{da^u}{dx} = a^u\cdot\frac{du}{dx}\cdot\log_e a.$

829. $\frac{dx^x}{dx} = x^x(1 + \log_e x).$

830. $\frac{d(\log_a x)}{dx} = \frac{1}{x\cdot\log_e a} = \frac{\log_a e}{x}$

831. $\frac{d\sin x}{dx} = \cos x.$

832. $\frac{d\cos x}{dx} = -\sin x.$

833. $\frac{d \tan x}{dx} = \sec^2 x.$

834. $\frac{d \operatorname{ctn} x}{dx} = -\csc^2 x.$

835. $\frac{d \sec x}{dx} = \tan x \cdot \sec x.$

836. $\frac{d \csc x}{dx} = -\operatorname{ctn} x \cdot \csc x.$

837. $\frac{d \sin^{-1} x}{dx} = \frac{1}{\sqrt{1 - x^2}}.$

838. $\frac{d \cos^{-1} x}{dx} = \frac{-1}{\sqrt{1 - x^2}}.$

839. $\frac{d \tan^{-1} x}{dx} = \frac{1}{1 + x^2}.$

840. $\frac{d \operatorname{ctn}^{-1} x}{dx} = -\frac{1}{1 + x^2}.$

841. $\frac{d \sec^{-1} x}{dx} = \frac{1}{x\sqrt{x^2 - 1}}.$

842. $\frac{d \csc^{-1} x}{dx} = -\frac{1}{x\sqrt{x^2 - 1}}.$

843. $\frac{d \sinh x}{dx} = \cosh x.$

844. $\frac{d \cosh x}{dx} = \sinh x.$

845. $\frac{d \tanh x}{dx} = \operatorname{sech}^2 x.$

846. $\frac{d \operatorname{ctnh} x}{dx} = -\operatorname{csch}^2 x.$

847. $$\frac{d \operatorname{sech} x}{dx} = -\operatorname{sech} x \cdot \tanh x.$$

848. $$\frac{d \operatorname{csch} x}{dx} = -\operatorname{csch} x \cdot \operatorname{ctnh} x.$$

849. $$\frac{d \sinh^{-1} x}{dx} = \frac{1}{\sqrt{x^2+1}}.$$

850. $$\frac{d \cosh^{-1} x}{dx} = \frac{1}{\sqrt{x^2-1}}.$$

851. $$\frac{d \tanh^{-1} x}{dx} = \frac{1}{1-x^2}.$$

852. $$\frac{d \operatorname{ctnh}^{-1} x}{dx} = \frac{1}{1-x^2}.$$

853. $$\frac{d \operatorname{sech}^{-1} x}{dx} = \frac{-1}{x\sqrt{1-x^2}}.$$

854. $$\frac{d \operatorname{csch}^{-1} x}{dx} = \frac{-1}{x\sqrt{x^2+1}}.$$

855. $$\frac{d}{db}\int_a^b f(x)\,dx = f(b).$$

856. $$\frac{d}{da}\int_a^b f(x)\,dx = -f(a).$$

857. $$\frac{d}{dc}\int_a^b f(x,c)\,dx = \int_a^b D_c f(x,c)\cdot dx + f(b,c)\frac{db}{dc} - f(a,c)\frac{da}{dc}.$$

858. $$\frac{d^n(u\cdot v)}{dx^n} = v\cdot\frac{d^n u}{dx^n} + n\cdot\frac{dv}{dx}\cdot\frac{d^{n-1}u}{dx^{n-1}}$$
$$+\frac{n(n-1)}{2!}\cdot\frac{d^2 v}{dx^2}\cdot\frac{d^{n-2}u}{dx^{n-2}} + \cdots + u\frac{d^n v}{dx^n}.$$

859. If $f(x, y, z, \cdots)$ is a homogeneous function of the nth order, so that $f(\lambda x, \lambda y, \lambda z, \cdots) \equiv \lambda^n f(x, y, z, \cdots)$,
$$x\cdot D_x f + y\cdot D_y f + z\cdot D_z f + \cdots \equiv nf.$$

860. If $x = \phi(y)$,

$$\frac{dy}{dx} = \frac{1}{\phi'(y)}, \quad \frac{d^2y}{dx^2} = -\frac{\phi''(y)}{[\phi'(y)]^3},$$

$$\frac{d^3y}{dx^3} = \frac{3[\phi''(y)]^2 - \phi'(y)\cdot\phi'''(y)}{[\phi'(y)]^5}.$$

861. If $x = f(t)$ and $y = \phi(t)$,

$$\frac{dy}{dx} = \frac{\phi'(t)}{f'(t)}, \quad \frac{d^2y}{dx^2} = \frac{f'(t)\cdot\phi''(t) - f''(t)\cdot\phi'(t)}{[f'(t)]^3}.$$

862. If $f(x, y) = 0$,

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y} \equiv -\frac{D_x f}{D_y f},$$

$$\frac{d^2y}{dx^2} = -\frac{D_x^2 f\cdot(D_y f)^2 - 2\,D_x D_y f\cdot D_x f\cdot D_y f + D_y^2 f\cdot(D_x f)^2}{(D_y f)^3}.$$

863. If $y = f(u, v)$, $u = \phi(x)$, and $v = \psi(x)$,

$$\frac{df}{dx} = \frac{\partial f}{\partial u}\cdot\frac{du}{dx} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dx} = u'\cdot D_u f + v'\cdot D_v f,$$

$$\frac{d^2f}{dx^2} = \frac{\partial^2 f}{\partial u^2}\cdot\left(\frac{du}{dx}\right)^2 + 2\,\frac{\partial^2 f}{\partial u\cdot\partial v}\cdot\frac{du}{dx}\cdot\frac{dv}{dx} + \frac{\partial^2 f}{\partial^2 v}\cdot\left(\frac{dv}{dx}\right)^2$$

$$+ \frac{\partial f}{\partial u}\cdot\frac{d^2u}{dx^2} + \frac{\partial f}{\partial v}\cdot\frac{d^2v}{dx^2}$$

$$= u'^2\cdot D^2_u f + 2\,u'\cdot v'\cdot D_u D_v f + v'^2\cdot D_v^2 f$$

$$+ u''\cdot D_u f + v''\cdot D_v f.$$

864. If $f(x, y, z) = 0$, $D_x z = -D_x f / D_z f$,

$$D_x^2 z = -[D_x^2 f\cdot(D_z f)^2$$

$$- 2\,D_z f\cdot D_x f\cdot D_x D_y f + D_z^2 f(D_x f)^2] / (D_z f)^3,$$

$$D_x D_y z = -[D_x D_y f\cdot(D_z f)^2 - D_z f\, D_x f\cdot D_y D_z f$$

$$+ D_z f\cdot D_y f\cdot D_x D_z f + D_x f\cdot D_y f\cdot D_z^2 f] / (D_z f)^3.$$

865. If $V = \phi(u, v)$, $u = f_1(x, y)$, and $v = f_2(x, y)$,

$$D_x V = D_u \phi \cdot D_x u + D_v \phi \cdot D_x v,$$

$$\begin{aligned} D_x^2 V = {} & D_u^2 \phi \cdot (D_x u)^2 + D_v^2 \phi \cdot (D_x v)^2 + 2\, D_u D_v \phi \,.\, D_x u \cdot D_x v \\ & + D_u \phi D_x^2 u + D_v \phi \,.\, D_x^2 v, \end{aligned}$$

$$\begin{aligned} D_y D_x V = {} & D_u^2 \phi \cdot D_x u \cdot D_y u + D_v^2 \phi \cdot D_x v \cdot D_y v \\ & + D_u D_v \phi (D_x v \cdot D_y u + D_x u \cdot D_y v) \\ & + D_u \phi \cdot D_x D_y u + D_v \phi \cdot D_x D_y v, \end{aligned}$$

$$\begin{aligned} D_x^2 V + D_y^2 V = {} & D_u^2 \phi \cdot [(D_x u)^2 + (D_y u)^2] \\ & + D_v^2 \phi \cdot [(D_x v)^2 + (D_y v)^2] \\ & + 2\, D_u D_v \phi \cdot [D_x u \cdot D_x v + D_y u \cdot D_y v] \\ & + D_u \phi \cdot [D_x^2 u + D_y^2 u] \\ & + D_v \phi \cdot [D_x^2 v + D_y^2 v]. \end{aligned}$$

In the special case, $u \equiv r \equiv \sqrt{x^2 + y^2}$, $v \equiv \theta \equiv \tan^{-1}(y / x)$, we have $D_r x = \cos\theta = x / \sqrt{x^2 + y^2}$; $D_r y = \sin\theta = y / \sqrt{x^2 + y^2}$;

$D_\theta x = -r \sin\theta = -y$; $D_\theta y = r \cos\theta = x$;

$D_x r = x / \sqrt{x^2 + y^2} = \cos\theta$; $D_y r = y / \sqrt{x^2 + y^2} = \sin\theta$;

$D_x \theta = -y / (x^2 + y^2) = -\sin\theta / r$;

$D_y \theta = x / (x^2 + y^2) = \cos\theta / r$; and

$$D_x^2 V + D_y^2 V = D_r^2 V + \frac{1}{r} \cdot D_r V + \frac{1}{r^2} \cdot D_\theta^2 V.$$

866. If $V = \phi(u, v)$, $u = f_1(r, \theta)$, and $v = f_2(r, \theta)$,

$$\begin{aligned} D_r^2 V + \frac{1}{r} \cdot D_r V + \frac{1}{r^2} \cdot D_\theta^2 V = {} & D_u^2 V \cdot \left[(D_r u)^2 + \frac{(D_\theta u)^2}{r^2} \right] \\ & + D_v^2 V \cdot \left[(D_r v)^2 + \frac{(D_\theta v)^2}{r^2} \right] \\ & + 2\, D_u D_v V \left[D_r u \cdot D_r v + \frac{D_\theta u \cdot D_\theta v}{r^2} \right] + \end{aligned}$$

$$+ D_u V \left[D_r^2 u + \frac{1}{r} \cdot D_r u + \frac{1}{r^2} \cdot D_\theta^2 u \right]$$

$$+ D_v V \left[D_r^2 v + \frac{1}{r} \cdot D_r v + \frac{1}{r^2} \cdot D_\theta^2 v \right].$$

867. If $V = \phi(u, v, w)$, $u = f_1(x, y, z)$, $v = f_2(x, y, z)$, and $w = f_3(x, y, z)$,

$$D_x V = D_u V \cdot D_x u + D_v V \cdot D_x v + D_w V \cdot D_x w,$$

$$\begin{aligned} D_x^2 V = {} & D_u^2 V \cdot (D_x u)^2 + D_v^2 V \cdot (D_x v)^2 + D_w^2 V \cdot (D_x w)^2 \\ & + D_u V \cdot D_x^2 u + D_v V \cdot D_x^2 v + D_w V \cdot D_x^2 w \\ & + 2\,(D_u D_v V \cdot D_x u \cdot D_x v + D_u D_w V \cdot D_x u \cdot D_x w \\ & + D_v D_w V \cdot D_x v \cdot D_x w). \end{aligned}$$

$$\begin{aligned} D_x^2 V + D_y^2 V + D_z^2 V = {} & D_u^2 V \cdot [(D_x u)^2 + (D_y u)^2 + (D_z u)^2] \\ & + D_v^2 V \cdot [(D_x v)^2 + (D_y v)^2 + (D_z v)^2] \\ & + D_w^2 V [(D_x w)^2 + (D_y w)^2 + (D_z w)^2] \\ & + 2\, D_u D_v V \cdot [D_x u \cdot D_x v + D_y u \cdot D_y v + D_z u \cdot D_z v] \\ & + 2\, D_v D_w V \cdot [D_x v \cdot D_x w + D_y v \cdot D_y w + D_z v \cdot D_z w] \\ & + 2\, D_w D_u V \cdot [D_x w \cdot D_x u + D_y w \cdot D_y u + D_z w \cdot D_z u] \\ & + D_u V \cdot [D_x^2 u + D_y^2 u + D_z^2 u] \\ & + D_v V \cdot [D_x^2 v + D_y^2 v + D_z^2 v] \\ & + D_w V \cdot [D_x^2 w + D_y^2 w + D_z^2 w]. \end{aligned}$$

In particular, if

$$x \equiv r \sin\theta \cos\phi, \quad y \equiv r \sin\theta \sin\phi, \quad z \equiv r \cos\theta,$$

so that $u \equiv r^2 \equiv x^2 + y^2 + z^2$, $v \equiv \theta \equiv \tan^{-1}(\sqrt{x^2 + y^2}/z)$, $w \equiv \phi \equiv \tan^{-1}(y/x)$, we have

$$D_r z = \cos\theta = z / \sqrt{x^2 + y^2 + z^2};$$

$$D_r x = \sin\theta \cos\phi = x / \sqrt{x^2 + y^2 + z^2};$$

$$D_r y = \sin\theta \sin\phi = y / \sqrt{x^2 + y^2 + z^2};$$

$$D_\theta z = -r \sin\theta = -\sqrt{x^2 + y^2};$$

$$D_\theta x = r \cos\theta \cos\phi = zx / \sqrt{x^2 + y^2};$$

$$D_\theta y = r \cos\theta \sin\phi = zy / \sqrt{x^2 + y^2};$$

$$D_\phi z = 0;$$

$$D_\phi x = -r \sin\theta \sin\phi = -y;$$

$$D_\phi y = r \sin\theta \cos\phi = x;$$

$$D_z r = z / r = \cos\theta;$$

$$D_z \theta = -\sqrt{x^2 + y^2} / r^2 = -\sin\theta / r;$$

$$D_z \phi = 0;$$

$$D_x r = x / r = \sin\theta \cos\phi;$$

$$D_x \theta = xz / r^2 \sqrt{x^2 + y^2} = \cos\theta \cos\phi / r;$$

$$D_x \phi = -y / (x^2 + y^2) = -\sin\phi / r \sin\theta;$$

$$D_y r = y / r = \sin\theta \sin\phi;$$

$$D_y \theta = zy / r^2 \sqrt{x^2 + y^2} = \cos\theta \sin\phi / r;$$

$$D_y \phi = x / (x^2 + y^2) = \cos\phi / r \sin\theta;$$

$$(D_x r)^2 + (D_y r)^2 + (D_z r)^2 = 1;$$

$$(D_x \theta)^2 + (D_y \theta)^2 + (D_z \theta)^2 = 1 / r^2;$$

$$(D_x \phi)^2 + (D_y \phi)^2 + (D_z \phi)^2 = 1 / r^2 \sin^2\theta;$$

$$(D_x V)^2 + (D_y V)^2 + (D_z V)^2 = (D_r V)^2 + \left(\frac{D_\theta V}{r}\right)^2 + \left(\frac{D_\phi V}{r \sin\theta}\right)^2;$$

$$D_x^2 V + D_y^2 V + D_z^2 V = \frac{1}{r^2 \sin\theta}\left[D_r(r^2 \cdot D_r V) \cdot \sin\theta + \frac{D_\phi^2 V}{\sin\theta} + D_\theta(\sin\theta \cdot D_\theta V)\right].$$

868. If $x = f_1(u, v)$, $y = f_2(u, v)$, $z = f_3(u, v)$,

$$D_x z = \frac{D_u f_3 \cdot D_v f_2 - D_v f_3 \cdot D_u f_2}{D_u f_1 \cdot D_v f_2 - D_v f_1 \cdot D_u f_2},$$

$$D_y z = \frac{D_v f_3 \cdot D_u f_1 - D_u f_3 \cdot D_v f_1}{D_u f_1 \cdot D_v f_2 - D_v f_1 \cdot D_u f_2}.$$

869. If $x = f(z, u)$, and $y = \phi(z, u)$,

$$D_x z = D_u \phi / (D_z f \cdot D_u \phi - D_z \phi \cdot D_u f),$$

$$D_y z = D_u f / (D_z \phi \cdot D_u f - D_z f \cdot D_u \phi).$$

870. If $F_1(x, y, z, u, v) = 0$,

$F_2(x, y, z, u, v) = 0$, and $F_3(x, y, z, u, v) = 0$,

$$D_x z \cdot \begin{vmatrix} D_z F_1 & D_u F_1 & D_v F_1 \\ D_z F_2 & D_u F_2 & D_v F_2 \\ D_z F_3 & D_u F_3 & D_v F_3 \end{vmatrix} = - \begin{vmatrix} D_x F_1 & D_u F_1 & D_v F_1 \\ D_x F_2 & D_u F_2 & D_v F_2 \\ D_x F_3 & D_u F_3 & D_v F_3 \end{vmatrix}.$$

871. If $F_1(x, y, z) = 0$, and $F_2(x, y, z) = 0$,

$$\frac{dy}{D_z F_1 \cdot D_x F_2 - D_z F_2 \cdot D_x F_1} = \frac{dz}{D_x F_1 \cdot D_y F_2 - D_x F_2 \cdot D_y F_1}$$

$$\frac{dx}{D_y F_1 \cdot D_z F_2 - D_y F_2 \cdot D_z F_1}.$$

If each of the quantities $y_1, y_2, y_3, \cdots y_n$ is a function of the n variables $x_1, x_2, x_3, \cdots x_n$, the determinant,

$$\begin{vmatrix} D_{x_1} y_1 & D_{x_2} y_1 & D_{x_3} y_1 \cdots & \\ D_{x_1} y_2 & D_{x_2} y_2 & D_{x_3} y_2 \cdots & \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ D_{x_1} y_n & D_{x_2} y_n & D_{x_3} y_n \cdots & D_{x_n} y_n \end{vmatrix}$$

is called the *functional determinant* or the *Jacobian* of the y's with respect to the x's and is denoted by the expression,

$$\frac{\partial(y_1, y_2, y_3, \cdots y_n)}{\partial(x_1, x_2, x_3, \cdots x_n)}, \text{ or by J } (y_1, y_2, \cdots y_n).$$

872. $$\frac{\partial(y_1, y_2, y_3, \cdots y_n)}{\partial(x_1, x_2, x_3, \cdots x_n)} \cdot \frac{\partial(x_1, x_2, x_3, \cdots x_n)}{\partial(y_1, y_2, y_3, \cdots y_n)} \equiv 1.$$

873. $$\frac{\partial(y_1, y_2, y_3, \cdots y_n)}{\partial(z_1, z_2, z_3, \cdots z_n)} \cdot \frac{\partial(z_1, z_2, z_3, \cdots z_n)}{\partial(x_1, x_2, x_3, \cdots x_n)} \equiv \frac{\partial(y_1, y_2, y_3, \cdots y_n)}{\partial(x_1, x_2, x_3, \cdots x_n)}.$$

If the y's are not all independent but are connected by an equation of the form $\phi(y_1, y_2, y_3, \cdots y_n) = 0$, the Jacobian of the y's with respect to the x's vanishes identically; and, conversely, if the Jacobian vanishes identically, the y's are connected by one or more relations of the above-mentioned form.

The *directional derivative* of any scalar point function, u, at any point, P, in any fixed direction PQ', is the limit, as PQ approaches zero, of the ratio of $u_Q - u_P$ to PQ, where Q is a point on the straight line PQ' between P and Q'. The *gradient*, h_u, of the function u at P is the directional derivative of u at P taken in the direction in which u increases most rapidly. This direction is normal to the surface of constant u which passes through P.

874. $h_u^2 \equiv (D_x u)^2 + (D_y u)^2 + (D_z u)^2.$

The directional derivative of any scalar point function at any point in any given direction is evidently equal to the product of the gradient and the cosine of the angle between the given direction and that in which the function increases most rapidly.

The *normal derivative*, at any point, P, of a point function u, taken with respect to another point function v, is the limit as PQ approaches zero of the ratio of $u_Q - u_P$ to $v_Q - v_P$, where Q is a point so chosen on the normal at P of the surface of constant v which passes through P, that $v_Q - v_P$ is positive. If (u, v) denotes the angle between the directions in which u and v increase most rapidly, the normal derivatives of u with respect to v, and of v with respect to u may be written

$$h_u \cos(u, v) \div h_v, \quad \text{and} \quad h_v \cdot \cos(u, v) \div h_u$$

respectively. If $h_u = h_v$, these derivatives are equal.

F. — MISCELLANEOUS FORMULAS.

If s is a plane analytic closed curve, n its normal drawn from within outwards, and dA the element of plane area within s, the usual integral transformation formulas for the functions u and v which, with their derivatives of the first order, are continuous everywhere within s, may be written —

875. $$\int u \cdot \cos(x, n)\, ds = \int\int D_x u \cdot dA.$$

876. $$\int [u \cdot \cos(x, n) + v \cdot \cos(y, n)]\, ds = \int\int (D_x u + D_y v)\, dA.$$

877. $$\int D_n u \cdot ds = \int\int (D_x^2 u + D_y^2 u)\, dA.$$

878. $$\begin{aligned} \int\int (D_x u \cdot D_x v + D_y u \cdot D_y v)\, dA \\ = \int u \cdot D_n v \cdot ds - \int\int u\,(D_x^2 v + D_y^2 v)\, dA \\ = \int v \cdot D_n u \cdot ds - \int\int v\,(D_x^2 u + D_y^2 v)\, dA. \end{aligned}$$

879. $$\begin{aligned} \int\int \lambda\,(D_x u \cdot D_x v + D_y u \cdot D_y v)\, dA = \int \lambda \cdot u \cdot D_n v \cdot ds \\ - \int\int u\,[D_x(\lambda \cdot D_x v) + D_y(\lambda \cdot D_y v)]\, dA. \end{aligned}$$

If ξ and η are two analytic functions which define a set of orthogonal curvilinear coördinates, and if (ξ, n) and (η, n) represent the angles between n and the directions in which ξ and η, respectively, increase most rapidly.

880. $$\int\int h_\xi \cdot h_\eta \cdot D_\eta\left(\frac{u}{h_\xi}\right) dA = \int u \cdot \cos(\eta, n)\, ds.$$

881. $$\int\int h_\xi \cdot h_\eta \cdot D_\xi\left(\frac{u}{h_\eta}\right) dA = \int u \cdot \cos(\xi, n)\, ds.$$

882. If r is the distance from a fixed point, Q, in the coördinate plane,

$$\int \frac{\cos(r, n)\, ds}{r} = 0,\ \pi, \text{ or } 2\pi,$$ according as Q is without, on, or within s.

If S is an analytic closed surface, n its normal drawn from within outwards, and $d\tau$ the element of volume shut in by S, the usual integral transformation formulas may be written —

883. $$\int\int u \cos(x, n)\, dS = \int\int\int D_x u \cdot d\tau.$$

884. $$\int\int [u \cos(x, n) + v \cos(y, n) + w \cos(z, n)]\, dS$$
$$= \int\int\int (D_x u + D_y v + D_z w)\, d\tau.$$

885. $$\int\int D_n u \cdot ds = \int\int\int (D_x^2 u + D_y^2 u + D_z^2 u)\, d\tau.$$

886. $$\int\int\int (D_x u \cdot D_x v + D_y u \cdot D_y v + D_z u \cdot D_z v)\, d\tau$$
$$= \int\int u \cdot D_n v \cdot dS - \int\int\int u\,(D_x^2 v + D_y^2 v + D_z^2 v)\, d\tau$$
$$= \int\int v \cdot D_n u \cdot dS - \int\int\int v\,(D_x^2 u + D_y^2 u + D_z^2 u)\, d\tau.$$

887. $$\int\int\int \lambda (D_x u \cdot D_x v + D_y u \cdot D_y v + D_z u \cdot D_z v)\, d\tau$$
$$= \int\int \lambda \cdot v \cdot D_n u \cdot dS$$
$$- \int\int\int v [D_x(\lambda D_x u) + D_y(\lambda D_y u) + D_z(\lambda D_z u)]\, d\tau.$$

If ξ, η, ζ are three analytic functions which define a system of orthogonal curvilinear coördinates,

888. $$\int\int\int h_\xi \cdot h_\eta \cdot h_\zeta \cdot D_\xi \left(\frac{u}{h_\eta \cdot h_\zeta}\right) d\tau = \int\int u \cdot \cos(\xi, n)\, dS.$$

889. $$\int\int\int h_\xi \cdot h_\eta \cdot h_\zeta \cdot D_\eta \left(\frac{u}{h_\xi \cdot h_\zeta}\right) d\tau = \int\int u \cdot \cos(\eta, n)\, dS.$$

890. $$\int\int\int h_\xi \cdot h_\eta \cdot h_\zeta \cdot D_\zeta \left(\frac{u}{h_\xi \cdot h_\eta}\right) d\tau = \int\int u \cdot \cos(\zeta, n)\, dS.$$

891. If r is the distance from a fixed point, Q,

$\int \frac{\cos(r, n)}{r^2}\, dS = 0, 2\pi$, or 4π according as Q is without, on, or within S.

Stokes's Theorem. — The line integral, taken around a closed curve, of the tangential component of a vector point function, is equal to the surface integral, taken over a surface bounded by the curve, of the normal component of the curl of the vector, the direction of integration around the curve forming a right-handed screw rotation about the normals.

If X, Y, Z are the components of the vector,

892. $$\int (X\,dx + Y\,dy + Z\,dz) = \int\int [(D_y Z - D_z Y) \cos(x, n)$$
$$+ (D_z X - D_x Z) \cos(y, n)$$
$$+ (D_x Y - D_y X) \cos(z, n)]\, dS.$$

Equations 893 to 897 give Poisson's Equation in orthogonal Cartesian, in cylindrical, in spherical, and in orthogonal curvilinear coördinates.

893. $$\bar{\nabla}^2 V \equiv D_x^2 V + D_y^2 V + D_z^2 V = -4\pi\rho.$$

894. $$\frac{1}{r} \cdot D_r(r \cdot D_r V) + \frac{1}{r^2} \cdot D_\theta^2 V + D_z^2 V = -4\pi\rho.$$

895. $$\sin\theta \cdot D_r(r^2 \cdot D_r V) + \frac{D_\phi^2 V}{\sin\theta} + D_\theta(\sin\theta \cdot D_\theta V) = -4\pi\rho r^2 \sin\theta.$$

896. $$h_\xi^2 \cdot D_\xi^2 V + h_\eta^2 \cdot D_\eta^2 V + h_\zeta^2 \cdot D_\zeta^2 V + D_\xi V \cdot \bar{\nabla}^2 \xi + D_\eta V \cdot \bar{\nabla}^2 \eta + D_\zeta V \cdot \bar{\nabla}^2 \zeta = -4\pi\rho.$$

897. $$h_\xi \cdot h_\eta \cdot h_\zeta \left\{ D_\xi\left(\frac{h_\xi}{h_\eta h_\zeta} \cdot D_\xi V\right) + D_\eta\left(\frac{h_\eta}{h_\xi h_\zeta} \cdot D_\eta V\right) + D_\zeta\left(\frac{h_\zeta}{h_\xi h_\eta} \cdot D_\zeta V\right)\right\} = -4\pi\rho$$

G. — CERTAIN CONSTANTS.

$\pi = 3.14159\ 26535\ 89793$

$\log_{10} \pi = 0.49714\ 98726\ 94134$

$\frac{1}{\pi} = 0.31830\ 98861\ 83791$

$\pi^2 = 9.86960\ 44010\ 89359$

$\sqrt{\pi} = 1.77245\ 38509\ 05516$

$\log_{10} 2 = 0.30102\ 99956\ 63981$

$e = 2.71828\ 18284\ 59045$

$\log_{10} e = 0.43429\ 44819\ 03252$

$\log_e 10 = 2.30258\ 50929\ 94046$

$\log_e 2 = 0.69314\ 71805\ 59945$

$\log_{10} \log_{10} e = 9.63778\ 43113\ 00537$

$\log_e \pi = 1.14472\ 98858\ 49400$

INTERPOLATION.

If values of an analytic function, $f(x)$, are given in a table for a number of values of the argument x, separated from one another consecutively by the constant small interval, δ, the differences between successive tabular values of the function are called *first tabular differences*, the differences of these first differences, *second tabular differences*, and so on. The tabular differences of the first, second, third, and fourth orders corresponding to $x = a$ are

$$\begin{aligned}
\Delta_1 &\equiv f(a + \delta) - f(a),\\
\Delta_2 &\equiv f(a + 2\,\delta) - 2 \cdot f(a + \delta) + f(a),\\
\Delta_3 &\equiv f(a + 3\,\delta) - 3 \cdot f(a + 2\,\delta) + 3 \cdot f(a + \delta) - f(a),\\
\Delta_4 &\equiv f(a + 4\,\delta) - 4 \cdot f(a + 3\,\delta) + 6 \cdot f(a + 2\,\delta) - 4 \cdot f(a + \delta) + f(a),
\end{aligned}$$

where $f(a)$ is any tabulated value.

The value of the function for $x = (a + h)$, where $h = k\delta$, is

$$f(a + h) = f(a) + k \cdot \Delta_1 + \frac{k(k-1)}{2!} \cdot \Delta_2 + \frac{k(k-1)(k-2)}{3!} \cdot \Delta_3 + \frac{k(k-1)(k-2)(k-3)}{4!} \cdot \Delta_4 + \cdots.$$

The Natural Logarithms (each increased by 10.) of Numbers between 0.00 and 0.99.

N.	0	1	2	3	4	5	6	7	8	9
0.0		5.395	6.088	6.493	6.781	7.004	7.187	7.341	7.474	7.592
0.1	7.697	7.793	7.880	7.960	8.034	8.103	8.167	8.228	8.285	8.339
0.2	8.391	8.439	8.486	8.530	8.573	8.614	8.653	8.691	8.727	8.762
0.3	8.796	8.829	8.861	8.891	8.921	8.950	8.978	9.006	9.032	9.058
0.4	9.084	9.108	9.132	9.156	9.179	9.201	9.223	9.245	9.266	9.287
0.5	9.307	9.327	9.346	9.365	9.384	9.402	9.420	9.438	9.455	9.472
0.6	9.489	9.506	9.522	9.538	9.554	9.569	9.584	9.600	9.614	9.629
0.7	9.643	9.658	9.671	9.685	9.699	9.712	9.726	9.739	9.752	9.764
0.8	9.777	9.789	9.802	9.814	9.826	9.837	9.849	9.861	9.872	9.883
0.9	9.895	9.906	9.917	9.927	9.938	9.949	9.959	9.970	9.980	9.990

$$\log_e x = \log_{10} x \cdot \log_e 10 = (2.30258) \log_{10} x.$$

The Natural Logarithms of Numbers between 1.0 and 9.9.

N.	0	1	2	3	4	5	6	7	8	9
1.	0.000	0.095	0.182	0.262	0.336	0.405	0.470	0.531	0.588	0.642
2.	0.693	0.742	0.788	0.833	0.875	0.916	0.956	0.993	1.030	1.065
3.	1.099	1.131	1.163	1.194	1.224	1.253	1.281	1.308	1.335	1.361
4.	1.386	1.411	1.435	1.459	1.482	1.504	1.526	1.548	1.569	1.589
5.	1.609	1.629	1.649	1.668	1.686	1.705	1.723	1.740	1.758	1.775
6.	1.792	1.808	1.825	1.841	1.856	1.872	1.887	1.902	1.917	1.932
7.	1.946	1.960	1.974	1.988	2.001	2.015	2.028	2.041	2.054	2.067
8.	2.079	2.092	2.104	2.116	2.128	2.140	2.152	2.163	2.175	2.186
9.	2.197	2.208	2.219	2.230	2.241	2.251	2.262	2.272	2.282	2.293

The Natural Logarithms of Whole Numbers from 10 to 109.

N.	0	1	2	3	4	5	6	7	8	9
1	2.303	2.398	2.485	2.565	2.639	2.708	2.773	2.833	2.890	2.944
2	2.996	3.045	3.091	3.135	3.178	3.219	3.258	3.296	3.332	3.367
3	3.401	3.434	3.466	3.497	3.526	3.555	3.584	3.611	3.638	3.664
4	3.689	3.714	3.738	3.761	3.784	3.807	3.829	3.850	3.871	3.892
5	3.912	3.932	3.951	3.970	3.989	4.007	4.025	4.043	4.060	4.078
6	4.094	4.111	4.127	4.143	4.159	4.174	4.190	4.205	4.220	4.234
7	4.248	4.263	4.277	4.290	4.304	4.317	4.331	4.344	4.357	4.369
8	4.382	4.394	4.407	4.419	4.431	4.443	4.454	4.466	4.477	4.489
9	4.500	4.511	4.522	4.533	4.543	4.554	4.564	4.575	4.585	4.595
10	4.605	4.615	4.625	4.635	4.644	4.654	4.663	4.673	4.682	4.691

The Values in Circular Measure of Angles which are given in Degrees and Minutes.

1′	0.0003	9′	0.0026	3°	0.0524	20°	0.3491	100°	1.7453
2′	0.0006	10′	0.0029	4°	0.0698	30°	0.5236	110°	1.9199
3′	0.0009	20′	0.0058	5°	0.0873	40°	0.6981	120°	2.0944
4′	0.0012	30′	0.0087	6°	0.1047	50°	0.8727	130°	2.2689
5′	0.0015	40′	0.0116	7°	0.1222	60°	1.0472	140°	2.4435
6′	0.0017	50′	0.0145	8°	0.1396	70°	1.2217	150°	2.6180
7′	0.0020	1°	0.0175	9°	0.1571	80°	1.3963	160°	2.7925
8′	0.0023	2°	0.0349	10°	0.1745	90°	1.5708	170°	2.9671

Equivalents of Radians in Degrees, Minutes, and Seconds of Arc.

Radians.	Equivalents.	Radians.	Equivalents.
0.0001	0° 0′ 20″.6	0.6000	34° 22′ 38″.9
0.0002	0° 0′ 41″.3	0.7000	40° 6′ 25″.4
0.0003	0° 1′ 01″.9	0.8000	45° 50′ 11″.8
0.0004	0° 1′ 22″.5	0.9000	51° 33′ 58″.3
0.0005	0° 1′ 43″.1	1.0000	57° 17′ 44″.8
0.0006	0° 2′ 03″.8	2.0000	114° 35′ 29″.6
0.0007	0° 2′ 24″.4	3.0000	171° 53′ 14″.4
0.0008	0° 2′ 45″.0	4.0000	229° 10′ 59″.2
0.0009	0° 3′ 05″.6	5.0000	286° 28′ 44″.0
0.0010	0° 3′ 26″.3	6.0000	343° 46′ 28″.8
0.0020	0° 6′ 52″.5	7.0000	401° 4′ 13″.6
0.0030	0° 10′ 18″.8	8.0000	458° 21′ 58″.4
0.0040	0° 13′ 45″.1	9.0000	515° 39′ 43″.3
0.0050	0° 17′ 11″.3	10.0000	572° 57′ 28″.1
0.0060	0° 20′ 37″.6	20.0000	1145° 54′ 56″.1
0.0070	0° 24′ 03″.9	30.0000	1718° 52′ 24″.2
0.0080	0° 27′ 30″.1	40.0000	2291° 49′ 52″.2
0.0090	0° 30′ 56″.4	50.0000	2864° 47′ 20″.3
0.0100	0° 34′ 22″.6	60.0000	3437° 44′ 48″.4
0.0200	1° 8′ 45″.3	70.0000	4010° 42′ 16″.4
0.0300	1° 43′ 07″.9	80.0000	4583° 39′ 44″.5
0.0400	2° 17′ 30″.6	90.0000	5156° 37′ 12″.6
0.0500	2° 51′ 53″.2	100.0000	5729° 34′ 40″.6
0.0600	3° 26′ 15″.9	$2\pi =$ 6.28319	360°
0.0700	4° 0′ 38″.5	$4\pi =$ 12.56637	720°
0.0800	4° 35′ 01″.2	$6\pi =$ 18.84956	1080°
0.0900	5° 9′ 23″.8	$8\pi =$ 25.13274	1440°
0.1000	5° 43′ 46″.5	$10\pi =$ 31.41593	1800°
0.2000	11° 27′ 33″.0	$12\pi =$ 37.69911	2160°
0.3000	17° 11′ 19″.4	$14\pi =$ 43.98230	2520°
0.4000	22° 55′ 05″.9	$16\pi =$ 50.26548	2880°
0.5000	28° 38′ 52″.4	$18\pi =$ 56.54867	3240°

The Square Roots of Certain Numbers between 0.0 and 11.

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	0.000	0.316	0.447	0.548	0.632	0.707	0.775	0.837	0.894	0.949
1	1.000	1.049	1.095	1.140	1.183	1.225	1.265	1.304	1.342	1.378
2	1.414	1.449	1.483	1.517	1.549	1.581	1.612	1.643	1.673	1.703
3	1.732	1.761	1.789	1.817	1.844	1.871	1.897	1.924	1.949	1.975
4	2.000	2.025	2.049	2.074	2.098	2.121	2.145	2.168	2.191	2.214
5	2.236	2.258	2.280	2.302	2.324	2.345	2.366	2.387	2.408	2.429
6	2.449	2.470	2.490	2.510	2.530	2.550	2.569	2.588	2.608	2.627
7	2.646	2.665	2.683	2.702	2.720	2.739	2.757	2.775	2.793	2.811
8	2.828	2.846	2.864	2.881	2.898	2.915	2.933	2.950	2.966	2.983
9	3.000	3.017	3.033	3.050	3.066	3.082	3.098	3.114	3.130	3.146
10	3.162	3.178	3.194	3.209	3.225	3.240	3.256	3.271	3.286	3.302

The Square Roots of Whole Numbers between 10 and 100.

	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
1	3.162	3.317	3.464	3.606	3.742	3.873	4.000	4.123	4.243	4.359
2	4.472	4.583	4.690	4.796	4.899	5.000	5.099	5.196	5.292	5.385
3	5.477	5.568	5.657	5.745	5.831	5.916	6.000	6.083	6.164	6.245
4	6.325	6.403	6.481	6.557	6.633	6.708	6.782	6.856	6.928	7.000
5	7.071	7.141	7.211	7.280	7.348	7.416	7.483	7.550	7.616	7.681
6	7.746	7.810	7.874	7.937	8.000	8.062	8.124	8.185	8.246	8.307
7	8.367	8.426	8.485	8.544	8.602	8.660	8.718	8.775	8.832	8.888
8	8.944	9.000	9.055	9.110	9.165	9.220	9.274	9.327	9.381	9.434
9	9.487	9.539	9.592	9.644	9.695	9.747	9.798	9.849	9.900	9.950

The Common Logarithms of e^x and e^{-x}.

x	$\log_{10} e^x$	$\log_{10} e^{-x}$
0.00001	0.0000043429	$\bar{1}.9999956571$
0.00002	0.0000086859	$\bar{1}.9999913141$
0.00003	0.0000130288	$\bar{1}.9999869712$
0.00004	0.0000173718	$\bar{1}.9999826282$
0.00005	0.0000217147	$\bar{1}.9999782853$
0.00006	0.0000260577	$\bar{1}.9999739423$
0.00007	0.0000304006	$\bar{1}.9999695994$
0.00008	0.0000347436	$\bar{1}.9999652564$
0.00009	0.0000390865	$\bar{1}.9999609135$
0.00010	0.0000434294	$\bar{1}.9999565706$
0.00020	0.0000868589	$\bar{1}.9999131411$
0.00030	0.0001302883	$\bar{1}.9998697117$
0.00040	0.0001737178	$\bar{1}.9998262822$
0.00050	0.0002171472	$\bar{1}.9997828528$
0.00060	0.0002605767	$\bar{1}.9997394233$
0.00070	0.0003040061	$\bar{1}.9996959939$
0.00080	0.0003474356	$\bar{1}.9996525644$
0.00090	0.0003908650	$\bar{1}.9996091350$
0.00100	0.0004342945	$\bar{1}.9995657055$
0.00200	0.0008685890	$\bar{1}.9991314110$
0.00300	0.0013028834	$\bar{1}.9986971166$
0.00400	0.0017371779	$\bar{1}.9982628221$
0.00500	0.0021714724	$\bar{1}.9978285276$
0.00600	0.0026057669	$\bar{1}.9973942331$
0.00700	0.0030400614	$\bar{1}.9969599386$
0.00800	0.0034743559	$\bar{1}.9965256441$
0.00900	0.0039086503	$\bar{1}.9960913497$
0.01000	0.0043429448	$\bar{1}.9956570552$
0.02000	0.0086858896	$\bar{1}.9913141104$
0.03000	0.0130288345	$\bar{1}.9869711655$
0.04000	0.0173717793	$\bar{1}.9826282207$
0.05000	0.0217147241	$\bar{1}.9782852759$
0.06000	0.0260576689	$\bar{1}.9739423311$
0.07000	0.0304006137	$\bar{1}.9695993863$

x	$\log_{10} e^{x}$	$\log_{10} e^{-x}$
0.08000	0.0347435586	$\bar{1}.9652564414$
0.09000	0.0390865034	$\bar{1}.9609134966$
0.10000	0.0434294482	$\bar{1}.9565705518$
0.20000	0.0868588964	$\bar{1}.9131411036$
0.30000	0.1302883446	$\bar{1}.8697116554$
0.40000	0.1737177928	$\bar{1}.8262822072$
0.50000	0.2171472410	$\bar{1}.7828527590$
0.60000	0.2605766891	$\bar{1}.7394233109$
0.70000	0.3040061373	$\bar{1}.6959938627$
0.80000	0.3474355855	$\bar{1}.6525644145$
0.90000	0.3908650337	$\bar{1}.6091349663$
1.00000	0.4342944819	$\bar{1}.5657055181$
2.00000	0.8685889638	$\bar{1}.1314110362$
3.00000	1.3028834457	$\bar{2}.6971165543$
4.00000	1.7371779276	$\bar{2}.2628220724$
5.00000	2.1714724095	$\bar{3}.8285275905$
6.00000	2.6057668914	$\bar{3}.3942331086$
7.00000	3.0400613733	$\bar{4}.9599386267$
8.00000	3.4743558552	$\bar{4}.5256441448$
9.00000	3.9086503371	$\bar{4}.0913496629$
10.00000	4.3429448190	$\bar{5}.6570551810$
20.00000	8.6858896381	$\bar{9}.3141103619$
30.00000	13.0288344571	$\overline{14}.9711655429$
40.00000	17.3717792761	$\overline{18}.6282207239$
50.00000	21.7147240952	$\overline{22}.2852759048$
60.00000	26.0576689142	$\overline{27}.9423310858$
70.00000	30.4006137332	$\overline{31}.5993862668$
80.00000	34.7435585523	$\overline{35}.2564414477$
90.00000	39.0865033713	$\overline{40}.9134966287$
100.00000	43.4294481903	$\overline{44}.5705518097$
200.00000	86.8588963807	$\overline{87}.1411036193$
300.00000	130.2883445710	$\overline{131}.7116554290$
400.00000	173.7177927613	$\overline{174}.2822072387$
500.00000	217.1472409516	$\overline{218}.8527590484$

Note: $\log e^{x+y} = \log e^{x} + \log e^{y}$. Thus, $\log e^{113.1478} = 49.139465180$.

The Values of e^{-x} for Certain Values of x.

x	$\log_{10} e^{-x}$	e^{-x}	x	$\log_{10} e^{-x}$	e^{-x}	x	$\log_{10} e^{-x}$	e^{-x}
1/10	9.956571	0.90484	9/5	9.218270	0.16530	25/4	7.285659	0.00193
1/8	9.945713	0.88250	2	9.131411	0.13533	32/5	7.220515	0.00167
1/6	9.927618	0.84648	9/4	9.022837	0.10540	7	6.959939	0.00091
1/5	9.913141	0.81873	5/2	8.914264	0.08209	36/5	6.873080	0.00075
1/4	9.891426	0.77880	8/3	8.841881	0.06948	8	6.525644	0.00034
1/3	9.855235	0.71653	3	8.697117	0.04979	81/10	6.482215	0.00030
2/5	9.826282	0.67032	25/8	8.642830	0.04394	49/6	6.453252	0.00028
1/2	9.782853	0.60653	16/5	8.610258	0.04076	25/3	6.380879	0.00024
2/3	9.710470	0.51342	18/5	8.436540	0.02732	9	6.091350	0.00012
4/5	9.652564	0.44933	4	8.262822	0.01832	49/5	5 743914	0.00006
9/10	9.609135	0.40657	25/6	8.190439	0.01550	10	5.657055	0.00004
1	9.565706	0.36788	9/2	8.045675	0.01111	32/3	5.367526	0.00002
9/8	9.511419	0.32465	49/10	7.871957	0.00745	11	5.222761	0.00002
4/3	9.420941	0.26360	5	7.828528	0.00674	12	4.788467	0 00001
3/2	9.348558	0.22313	6	7.394233	0.00248	13	4.354173	0.00000
8/5	9.305129	0.20190	49/8	7.339946	0.00218	14	3.919877	0.00000

These quantities with the numbers in the preceding table are useful in computing the values of series of the form

$$\sum_{k=1}^{k=\infty} A_k \cdot e^{-k^2 mt}.$$

Values of the Complete Elliptic Integrals, K and E, for Different Values of the Modulus, k.

$\sin^{-1} k$	K	E	$\sin^{-1} k$	K	E	$\sin^{-1} k$	K	E
0°	1.5708	1.5708	30°	1.6858	1.4675	60°	2.1565	1.2111
1°	1.5709	1.5707	31°	1.6941	1.4608	61°	2.1842	1.2015
2°	1.5713	1.5703	32°	1.7028	1.4539	62°	2.2132	1.1920
3°	1.5719	1.5697	33°	1.7119	1.4469	63°	2.2435	1.1826
4°	1.5727	1.5689	34°	1.7214	1.4397	64°	2.2754	1.1732
5°	1.5738	1.5678	35°	1.7312	1.4323	65°	2.3088	1.1638
6°	1.5751	1.5665	36°	1.7415	1.4248	66°	2.3439	1.1545
7°	1.5767	1.5649	37°	1.7522	1.4171	67°	2.3809	1.1453
8°	1.5785	1.5632	38°	1.7633	1.4092	68°	2.4198	1.1362
9°	1.5805	1.5611	39°	1.7748	1.4013	69°	2.4610	1.1272
10°	1.5828	1.5589	40°	1.7868	1.3931	70°	2.5046	1.1184
11°	1.5854	1.5564	41°	1.7992	1.3849	71°	2.5507	1.1096
12°	1.5882	1.5537	42°	1.8122	1.3765	72°	2.5998	1.1011
13°	1.5913	1.5507	43°	1.8256	1.3680	73°	2.6521	1.0927
14°	1.5946	1.5476	44°	1.8396	1.3594	74°	2.7081	1.0844
15°	1.5981	1.5442	45°	1.8541	1.3506	75°	2.7681	1.0764
16°	1.6020	1.5405	46°	1.8691	1.3418	76°	2.8327	1.0686
17°	1.6061	1.5367	47°	1.8848	1.3329	77°	2.9026	1.0611
18°	1.6105	1.5326	48°	1.9011	1.3238	78°	2.9786	1.0538
19°	1.6151	1.5283	49°	1.9180	1.3147	79°	3.0617	1.0468
20°	1.6200	1.5238	50°	1.9356	1.3055	80°	3.1534	1.0401
21°	1.6252	1.5191	51°	1.9539	1.2963	81°	3.2553	1.0338
22°	1.6307	1.5141	52°	1.9729	1.2870	82°	3.3699	1.0278
23°	1.6365	1.5090	53°	1.9927	1.2776	83°	3.5004	1.0223
24°	1.6426	1.5037	54°	2.0133	1.2681	84°	3.6519	1.0172
25°	1.6490	1.4981	55°	2.0347	1.2587	85°	3.8317	1.0127
26°	1.6557	1.4924	56°	2.0571	1.2492	86°	4.0528	1.0086
27°	1.6627	1.4864	57°	2.0804	1.2397	87°	4.3387	1.0053
28°	1.6701	1.4803	58°	2.1047	1.2301	88°	4.7427	1.0026
29°	1.6777	1.4740	59°	2.1300	1.2206	89°	5.4349	1.0008

Values of $F(k, \phi)$ for Certain Values of k and ϕ.

$$F(k, \phi) = \int_0^\phi \frac{dz}{\sqrt{1 - k^2 \sin^2 z}}.$$

ϕ	$\alpha = \sin^{-1} k.$								
	0°	10°	15°	30°	45°	60°	75°	80°	90°
1°	0.0174	0.0174	0.0174	0.0174	0.0174	0.0174	0.0174	0.0174	0.0174
2°	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349
3°	0.0524	0.0524	0.0524	0.0524	0.0524	0.0524	0.0524	0.0524	0.0524
4°	0.0698	0.0698	0.0698	0.0698	0.0698	0.0699	0.0699	0.0699	0.0699
5°	0.0873	0.0873	0.0873	0.0873	0.0873	0.0874	0.0874	0.0874	0.0874
10°	0.1745	0.1746	0.1746	0.1748	0.1750	0.1752	0.1754	0.1754	0.1754
15°	0.2618	0.2619	0.2620	0.2625	0.2633	0.2641	0.2646	0.2647	0.2648
20°	0.3491	0.3493	0.3495	0.3508	0.3526	0.3545	0.3559	0.3562	0.3564
25°	0.4363	0.4367	0.4372	0.4397	0.4433	0.4470	0.4498	0.4504	0.4509
30°	0.5236	0.5243	0.5251	0.5294	0.5356	0.5422	0.5474	0.5484	0.5493
35°	0.6109	0.6119	0.6132	0.6200	0.6300	0.6408	0.6495	0.6513	0.6528
40°	0.6981	0.6997	0.7016	0.7116	0.7267	0.7436	0.7574	0.7604	0.7629
45°	0.7854	0.7876	0.7902	0.8044	0.8260	0.8512	0.8727	0.8774	0.8814
50°	0.8727	0.8756	0.8792	0.8982	0.9283	0.9646	0.9971	1.0044	1.0107
55°	0.9599	0.9637	0.9683	0.9933	1.0337	1.0848	1.1331	1.1444	1.1542
60°	1.0472	1.0519	1.0577	1.0896	1.1424	1.2125	1.2837	1.3014	1.3170
65°	1.1345	1.1402	1.1474	1.1869	1.2545	1.3489	1.4532	1.4810	1.5064
70°	1.2217	1.2286	1.2373	1.2853	1.3697	1.4944	1.6468	1.6918	1.7354
75°	1.3090	1.3171	1.3273	1.3846	1.4879	1.6492	1.8714	1.9468	2.0276
80°	1.3963	1.4056	1.4175	1.4846	1.6085	1.8125	2.1339	2.2653	2.4362
85°	1.4835	1.4942	1.5078	1.5850	1.7308	1.9826	2.4366	2.6694	3.1313
86°	1.5010	1.5120	1.5259	1.6052	1.7554	2.0172	2.5013	2.7612	3.3547
87°	1.5184	1.5297	1.5439	1.6253	1.7801	2.0519	2.5670	2.8561	3.6425
88°	1.5359	1.5474	1.5620	1.6454	1.8047	2.0867	2.6336	2.9537	4.0481
89°	1.5533	1.5651	1.5801	1.6656	1.8294	2.1216	2.7007	3.0530	4.7414
90°	1.5708	1.5828	1.5981	1.6858	1.8541	2.1565	2.7681	3.1534	Inf.

Values of $E(k, \phi)$ for Certain Values of k and ϕ.

$$E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 z} \cdot dz.$$

ϕ	$\alpha = \sin^{-1} k$.								
	0°	10°	15°	30°	45	60°	75°	80°	90°
1°	0.0174	0.0174	0.0174	0.0174	0.0174	0.0174	0.0174	0.0174	0.0174
2°	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349
3°	0.0524	0.0524	0.0524	0.0524	0.0524	0.0523	0.0523	0.0523	0.0523
4°	0.0698	0.0698	0.0698	0.0698	0.0698	0.0698	0.0698	0.0698	0.0698
5°	0.0873	0.0873	0.0873	0.0872	0.0872	0.0872	0.0872	0.0872	0.0872
10°	0.1745	0.1745	0.1745	0.1743	0.1741	0.1739	0.1737	0.1737	0.1736
15°	0.2618	0.2617	0.2616	0.2611	0.2603	0.2596	0.2590	0.2589	0.2588
20°	0.3491	0.3489	0.3486	0.3473	0.3456	0.3438	0.3425	0.3422	0.3420
25°	0.4363	0.4359	0.4354	0.4330	0.4296	0.4261	0.4236	0.4230	0.4226
30°	0.5236	0.5229	0.5221	0.5179	0.5120	0.5061	0.5016	0.5007	0.5000
35°	0.6109	0.6098	0.6085	0.6019	0.5928	0.5833	0.5762	0.5748	0.5736
40°	0.6981	0.6966	0.6947	0.6851	0.6715	0.6575	0.6468	0.6446	0.6428
45°	0.7854	0.7832	0.7806	0.7672	0.7482	0.7282	0.7129	0.7097	0.7071
50°	0.8727	0.8698	0.8663	0.8483	0.8226	0.7954	0.7741	0.7697	0.7660
55°	0.9599	0.9562	0.9517	0.9284	0.8949	0.8588	0.8302	0.8242	0.8192
60°	1.0472	1.0426	1.0368	1.0076	0.9650	0.9184	0.8808	0.8728	0.8660
65°	1.1345	1.1288	1.1218	1.0858	1.0329	0.9743	0.9258	0.9152	0.9063
70°	1.2217	1.2149	1.2065	1.1632	1.0990	1.0266	0.9652	0.9514	0.9397
75°	1.3090	1.3010	1.2911	1.2399	1.1635	1.0759	0.9992	0.9814	0.9659
80°	1.3963	1.3870	1.3755	1.3161	1.2266	1.1225	1.0282	1.0054	0.9848
85°	1.4835	1.4729	1.4598	1.3919	1.2889	1.1673	1.0534	1.0244	0.9962
86°	1.5010	1.4901	1.4767	1.4070	1.3012	1.1761	1.0581	1.0277	0.9976
87°	1.5184	1.5073	1.4936	1.4221	1.3136	1.1848	1.0628	1.0309	0.9986
88°	1.5359	1.5245	1.5104	1.4372	1.3260	1.1936	1.0674	1.0340	0.9994
89°	1.5533	1.5417	1.5273	1.4524	1.3383	1.2023	1.0719	1.0371	0.9998
90°	1.5708	1.5589	1.5442	1.4675	1.3506	1.2111	1.0764	1.0401	1.0000

Hyperbolic Functions.

x	e^x	e^{-x}	sinh x	cosh x	gd x
0.00	1.0000	1.0000	0.0000	1.0000	0°.0000
.01	1.0100	0.9900	.0100	1.0000	0.5729
.02	1.0202	.9802	.0200	1.0002	1.1458
.03	1.0305	.9704	.0300	1.0004	1.7186
.04	1.0408	.9608	.0400	1.0008	2.2912
.05	1.0513	.9512	.0500	1.0013	2.8636
.06	1.0618	.9418	.0600	1.0018	3.4357
.07	1.0725	.9324	.0701	1.0025	4.0074
.08	1.0833	.9231	.0801	1.0032	4.5788
.09	1.0942	.9139	.0901	1.0041	5.1497
.10	1.1052	.9048	.1002	1.0050	5.720
.11	1.1163	.8958	.1102	1.0061	6.290
.12	1.1275	.8869	.1203	1.0072	6.859
.13	1.1388	.8781	.1304	1.0085	7.428
.14	1.1503	.8694	.1405	1.0098	7.995
.15	1.1618	.8607	.1506	1.0113	8.562
.16	1.1735	.8521	.1607	1.0128	9.128
.17	1.1853	.8437	.1708	1.0145	9.694
.18	1.1972	.8353	.1810	1.0162	10.258
.19	1.2092	.8270	.1911	1.0181	10.821
.20	1.2214	.8187	.2013	1.0201	11.384
.21	1.2337	.8106	.2115	1.0221	11.945
.22	1.2461	.8025	.2218	1.0243	12.505
.23	1.2586	.7945	.2320	1.0266	13.063
.24	1.2712	.7866	.2423	1.0289	13.621
.25	1.2840	.7788	.2526	1.0314	14.177
.26	1.2969	.7711	.2629	1.0340	14.732
.27	1.3100	.7634	.2733	1.0367	15.285
.28	1.3231	.7558	.2837	1.0395	15.837
.29	1.3364	.7483	.2941	1.0423	16.388
.30	1.3499	.7408	.3045	1.0453	16.937
.31	1.3634	.7334	.3150	1.0484	17.484
.32	1.3771	.7261	.3255	1.0516	18.030
.33	1.3910	.7189	.3360	1.0549	18.573
.34	1.4049	.7118	.3466	1.0584	19.116
.35	1.4191	.7047	.3572	1.0619	19.656
.36	1.4333	.6977	.3678	1.0655	20.195
.37	1.4477	.6907	.3785	1.0692	20.732
.38	1.4623	.6839	.3892	1.0731	21.267
.39	1.4770	.6771	.4000	1.0770	21.800
.40	1.4918	.6703	.4108	1.0811	22.331
.41	1.5068	.6636	.4216	1.0852	22.859
.42	1.5220	.6570	.4325	1.0895	23.386
.43	1.5373	.6505	.4434	1.0939	23.911
.44	1.5527	.6440	.4543	1.0984	24.434
.45	1.5683	.6376	.4653	1.1030	24.955
.46	1.5841	.6313	.4764	1.1077	25.473
.47	1.6000	.6250	.4875	1.1125	25.989
.48	1.6161	.6188	.4986	1.1174	26.503
.49	1.6323	.6126	.5098	1.1225	27.015
0.50	1.6487	0.6065	0.5211	1.1276	27°.524

NOTE.—This table is taken from Prof. Byerly's Treatise on Fourier's Series, published by Messrs. Ginn & Co.

Hyperbolic Functions.

x	e^x	e^{-x}	$\sinh x$	$\cosh x$	$\operatorname{gd} x$
0.50	1.6487	0.6065	0.5211	1.1276	27°.524
.51	1.6653	.6005	.5324	1.1329	28.031
.52	1.6820	.5945	.5438	1.1383	28.535
.53	1.6989	.5886	.5552	1.1438	29.037
.54	1.7160	.5827	.5666	1.1494	29.537
.55	1.7333	.5770	.5782	1.1551	30.034
.56	1.7507	.5712	.5897	1.1609	30.529
.57	1.7683	.5655	.6014	1.1669	31.021
.58	1.7860	.5599	.6131	1.1730	31.511
.59	1.8040	.5543	.6248	1.1792	31.998
.60	1.8221	.5488	.6367	1.1855	32.483
.61	1.8404	.5433	.6485	1.1919	32.965
.62	1.8589	.5379	.6605	1.1984	33.444
.63	1.8776	.5326	.6725	1.2051	33.921
.64	1.8965	.5273	.6846	1.2119	34.395
.65	1.9155	.5220	.6967	1.2188	34.867
.66	1.9348	.5169	.7090	1.2258	35.336
.67	1.9542	.5117	.7213	1.2330	35.802
.68	1.9739	.5066	.7336	1.2402	36.265
.69	1.9937	.5016	.7461	1.2476	36.726
.70	2.0138	.4966	.7586	1.2552	37.183
.71	2.0340	.4916	.7712	1.2628	37.638
.72	2.0544	.4867	.7838	1.2706	38.091
.73	2.0751	.4819	.7966	1.2785	38.540
.74	2.0959	.4771	.8094	1.2865	38.987
.75	2.1170	.4724	.8223	1.2947	39.431
.76	2.1383	.4677	.8353	1.3030	39.872
.77	2.1598	.4630	.8484	1.3114	40.310
.78	2.1815	.4584	.8615	1.3199	40.746
.79	2.2034	.4538	.8748	1.3286	41.179
.80	2.2255	.4493	.8881	1.3374	41.608
.81	2.2479	.4449	.9015	1.3464	42.035
.82	2.2705	.4404	.9150	1.3555	42.460
.83	2.2933	.4360	.9286	1.3647	42.881
.84	2.3164	.4317	.9423	1.3740	43.299
.85	2.3396	.4274	.9561	1.3835	43.715
.86	2.3632	.4232	.9700	1.3932	44.128
.87	2.3869	.4190	.9840	1.4029	44.537
.88	2.4109	.4148	.9981	1.4128	44.944
.89	2.4351	.4107	1.0122	1.4229	45.348
.90	2.4596	.4066	1.0265	1.4331	45.750
.91	2.4843	.4025	1.0409	1.4434	46.148
.92	2.5093	.3985	1.0554	1.4539	46.544
.93	2.5345	.3946	1.0700	1.4645	46.936
.94	2.5600	.3906	1.0847	1.4753	47.326
.95	2.5857	.3867	1.0995	1.4862	47.713
.96	2.6117	.3829	1.1144	1.4973	48.097
.97	2.6379	.3791	1.1294	1.5085	48.478
.98	2.6645	.3753	1.1446	1.5199	48.857
.99	2.6912	.3716	1.1598	1.5314	49.232
1.00	2.7183	0.3679	1.1752	1.5431	49°.605

Hyperbolic Functions.

x	$l \sinh x$	$l \cosh x$	x	$l \sinh x$	$l \cosh x$	x	$l \sinh x$	$l \cosh x$
1.00	0.0701	0.1884	1.50	0.3282	0.3715	2.00	0.5595	0.5754
1.01	.0758	.1917	1.51	.3330	.3754	2.01	.5640	.5796
1.02	.0815	.1950	1.52	.3378	.3794	2.02	.5685	.5838
1.03	.0871	.1984	1.53	.3426	.3833	2.03	.5730	.5880
1.04	.0927	.2018	1.54	.3474	.3873	2.04	.5775	.5922
1.05	.0982	.2051	1.55	.3521	.3913	2.05	.5820	.5964
1.06	.1038	.2086	1.56	.3569	.3952	2.06	.5865	.6006
1.07	.1093	.2120	1.57	.3616	.3992	2.07	.5910	.6048
1.08	.1148	.2154	1.58	.3663	.4032	2.08	.5955	.6090
1.09	.1203	.2189	1.59	.3711	.4072	2.09	.6000	.6132
1.10	.1257	.2223	1.60	.3758	.4112	2.10	.6044	.6175
1.11	.1311	.2258	1.61	.3805	.4152	2.11	.6089	.6217
1.12	.1365	.2293	1.62	.3852	.4192	2.12	.6134	.6259
1.13	.1419	.2328	1.63	.3899	.4232	2.13	.6178	.6301
1.14	.1472	.2364	1.64	.3946	.4273	2.14	.6223	.6343
1.15	.1525	.2399	1.65	.3992	.4313	2.15	.6268	.6386
1.16	.1578	.2435	1.66	.4039	.4353	2.16	.6312	.6428
1.17	.1631	.2470	1.67	.4086	.4394	2.17	.6357	.6470
1.18	.1684	.2506	1.68	.4132	.4434	2.18	.6401	.6512
1.19	.1736	.2542	1.69	.4179	.4475	2.19	.6446	.6555
1.20	.1788	.2578	1.70	.4225	.4515	2.20	.6491	.6597
1.21	.1840	.2615	1.71	.4272	.4556	2.21	.6535	.6640
1.22	.1892	.2651	1.72	.4318	.4597	2.22	.6580	.6682
1.23	.1944	.2688	1.73	.4364	.4637	2.23	.6624	.6724
1.24	.1995	.2724	1.74	.4411	.4678	2.24	.6668	.6767
1.25	.2046	.2761	1.75	.4457	.4719	2.25	.6713	.6809
1.26	.2098	.2798	1.76	.4503	.4760	2.26	.6757	.6852
1.27	.2148	.2835	1.77	.4549	.4801	2.27	.6802	.6894
1.28	.2199	.2872	1.78	.4595	.4842	2.28	.6846	.6937
1.29	.2250	.2909	1.79	.4641	.4883	2.29	.6890	.6979
1.30	.2300	.2947	1.80	.4687	.4924	2.30	.6935	.7022
1.31	.2351	.2984	1.81	.4733	.4965	2.31	.6979	.7064
1.32	.2401	.3022	1.82	.4778	.5006	2.32	.7023	.7107
1.33	.2451	.3059	1.83	.4824	.5048	2.33	.7067	.7150
1.34	.2501	.3097	1.84	.4870	.5089	2.34	.7112	.7192
1.35	.2551	.3135	1.85	.4915	.5130	2.35	.7156	.7235
1.36	.2600	.3173	1.86	.4961	.5172	2.36	.7200	.7278
1.37	.2650	.3211	1.87	.5007	.5213	2.37	.7244	.7320
1.38	.2699	.3249	1.88	.5052	.5254	2.38	.7289	.7363
1.39	.2748	.3288	1.89	.5098	.5296	2.38	.7333	.7406
1.40	.2797	.3326	1.90	.5143	.5337	2.40	.7377	.7448
1.41	.2846	.3365	1.91	.5188	.5379	2.41	.7421	.7491
1.42	.2895	.3403	1.92	.5234	.5421	2.42	.7465	.7534
1.43	.2944	.3442	1.93	.5279	.5462	2.43	.7509	.7577
1.44	.2993	.3481	1.94	.5324	.5504	2.44	.7553	.7619
1.45	.3041	.3520	1.95	.5370	.5545	2.45	.7597	.7662
1.46	.3090	.3559	1.96	.5415	.5687	2.46	.7642	.7705
1.47	.3138	.3598	1.97	.5460	.5629	2.47	.7686	.7748
1.48	.3186	.3637	1.98	.5505	.5671	2.48	.7730	.7791
1.49	.3234	.3676	1.99	.5550	.5713	2.49	.7774	.7833
1.50	0.3282	0.3715	2.00	0.5595	0.5754	2.50	0.7818	0.7876

Hyperbolic Functions.

x	$l \sinh x$	$l \cosh x$	x	$l \sinh x$	$l \cosh x$	x	$l \sinh x$	$l \cosh x$
2.50	0.7818	0.7876	2.75	0.8915	0.8951	3.0	1.0008	1.0029
2.51	.7862	.7919	2.76	.8959	.8994	3.1	1.0444	1.0462
2.52	.7906	.7962	2.77	.9003	.9037	3.2	1.0880	1.0894
2.53	.7950	.8005	2.78	.9046	.9080	3.3	1.1316	1.1327
2.54	.7994	.8048	2.79	.9090	.9123	3.4	1.1751	1.1761
2.55	.8038	.8091	2.80	.9134	.9166	3.5	1.2186	1.2194
2.56	.8082	.8134	2.81	.9178	.9209	3.6	1.2621	1.2628
2.57	.8126	.8176	2.82	.9221	.9252	3.7	1.3056	1.3061
2.58	.8169	.8219	2.83	.9265	.9295	3.8	1.3491	1.3495
2.59	.8213	.8262	2.84	.9309	.9338	3.9	1.3925	1.3929
2.60	.8257	.8305	2.85	.9353	.9382	4.0	1.4360	1.4363
2.61	.8301	.8348	2.86	.9396	.9425	4.1	1.4795	1.4797
2.62	.8345	.8391	2.87	.9440	.9468	4.2	1.5229	1.5231
2.63	.8389	.8434	2.88	.9484	.9511	4.3	1.5664	1.5665
2.64	.8433	.8477	2.89	.9527	.9554	4.4	1.6098	1.6099
2.65	.8477	.8520	2.90	.9571	.9597	4.5	1.6532	1.6533
2.66	.8521	.8563	2.91	.9615	.9641	4.6	1.6967	1.6968
2.67	.8564	.8606	2.92	.9658	.9684	4.7	1.7401	1.7402
2.68	.8608	.8649	2.93	.9702	.9727	4.8	1.7836	1.7836
2.69	.8652	.8692	2.94	.9746	.9770	4.9	1.8270	1.8270
2.70	.8696	.8735	2.95	.9789	.9813	5.0	1.8704	1.8705
2.71	.8740	.8778	2.96	.9833	.9856	6.0	2.3047	2.3047
2.72	.8784	.8821	2.97	.9877	.9900	7.0	2.7390	2.7390
2.73	.8827	.8864	2.98	.9920	.9943	8.0	3.1733	3.1733
2.74	.8871	.8907	2.99	.9964	.9986	9.0	3.6076	3.6076
2.75	0.8915	0.8951	3.00	1.0008	1.0029	10.0	4.0419	4.0419

For values of x greater than 7.0, we may write, to five places of decimals at least,

$$\log_{10} \sinh x = \log_{10} \cosh x = \log \tfrac{1}{2} e^x = x\,(0.4342945) + \bar{1}.6989700.$$

The Common Logarithms of $\Gamma(n)$ for Values of n between 1 and 2.

n	$\log_{10}\Gamma(n)$	n	$\log_{10}\Gamma(n)$	n	$\log_{10}\Gamma(n)$	n	$\log_{10}\Gamma(n)$	n	$\log_{10}\Gamma(n)$
1.01	$\bar{1}.9975$	1.21	$\bar{1}.9617$	1.41	$\bar{1}.9478$	1.61	$\bar{1}.9517$	1.81	$\bar{1}.9704$
1.02	$\bar{1}.9951$	1.22	$\bar{1}.9605$	1.42	$\bar{1}.9476$	1.62	$\bar{1}.9523$	1.82	$\bar{1}.9717$
1.03	$\bar{1}.9928$	1.23	$\bar{1}.9594$	1.43	$\bar{1}.9475$	1.63	$\bar{1}.9529$	1.83	$\bar{1}.9730$
1.04	$\bar{1}.9905$	1.24	$\bar{1}.9583$	1.44	$\bar{1}.9473$	1.64	$\bar{1}.9536$	1.84	$\bar{1}.9743$
1.05	$\bar{1}.9883$	1.25	$\bar{1}.9573$	1.45	$\bar{1}.9473$	1.65	$\bar{1}.9543$	1.85	$\bar{1}.9757$
1.06	$\bar{1}.9862$	1.26	$\bar{1}.9564$	1.46	$\bar{1}.9472$	1.66	$\bar{1}.9550$	1.86	$\bar{1}.9771$
1.07	$\bar{1}.9841$	1.27	$\bar{1}.9554$	1.47	$\bar{1}.9473$	1.67	$\bar{1}.9558$	1.87	$\bar{1}.9786$
1.08	$\bar{1}.9821$	1.28	$\bar{1}.9546$	1.48	$\bar{1}.9473$	1.68	$\bar{1}.9566$	1.88	$\bar{1}.9800$
1.09	$\bar{1}.9802$	1.29	$\bar{1}.9538$	1.49	$\bar{1}.9474$	1.69	$\bar{1}.9575$	1.89	$\bar{1}.9815$
1.10	$\bar{1}.9783$	1.30	$\bar{1}.9530$	1.50	$\bar{1}.9475$	1.70	$\bar{1}.9584$	1.90	$\bar{1}.9831$
1.11	$\bar{1}.9765$	1.31	$\bar{1}.9523$	1.51	$\bar{1}.9477$	1.71	$\bar{1}.9593$	1.91	$\bar{1}.9846$
1.12	$\bar{1}.9748$	1.32	$\bar{1}.9516$	1.52	$\bar{1}.9479$	1.72	$\bar{1}.9603$	1.92	$\bar{1}.9862$
1.13	$\bar{1}.9731$	1.33	$\bar{1}.9510$	1.53	$\bar{1}.9482$	1.73	$\bar{1}.9613$	1.93	$\bar{1}.9878$
1.14	$\bar{1}.9715$	1.34	$\bar{1}.9505$	1.54	$\bar{1}.9485$	1.74	$\bar{1}.9623$	1.94	$\bar{1}.9895$
1.15	$\bar{1}.9699$	1.35	$\bar{1}.9500$	1.55	$\bar{1}.9488$	1.75	$\bar{1}.9633$	1.95	$\bar{1}.9912$
1.16	$\bar{1}.9684$	1.36	$\bar{1}.9495$	1.56	$\bar{1}.9492$	1.76	$\bar{1}.9644$	1.96	$\bar{1}.9929$
1.17	$\bar{1}.9669$	1.37	$\bar{1}.9491$	1.57	$\bar{1}.9496$	1.77	$\bar{1}.9656$	1.97	$\bar{1}.9946$
1.18	$\bar{1}.9655$	1.38	$\bar{1}.9487$	1.58	$\bar{1}.9501$	1.78	$\bar{1}.9667$	1.98	$\bar{1}.9964$
1.19	$\bar{1}.9642$	1.39	$\bar{1}.9483$	1.59	$\bar{1}.9506$	1.79	$\bar{1}.9679$	1.99	$\bar{1}.9982$
1.20	$\bar{1}.9629$	1.40	$\bar{1}.9481$	1.60	$\bar{1}.9511$	1.80	$\bar{1}.9691$	2.00	0.0000

$$\Gamma(z+1) = z \cdot \Gamma(z), \quad z > 1.$$

NATURAL TRIGONOMETRIC FUNCTIONS.

Angle.	Sin.	Csc.	Tan.	Ctn.	Sec.	Cos.	
0°	0.000	∞	0.000	∞	1.000	1.000	90°
1	0.017	57.30	0.017	57.29	1.000	1.000	89
2	0.035	28.65	0.035	28.64	1.001	0.999	88
3	0.052	19.11	0.052	19.08	1.001	0.999	87
4	0.070	14.34	0.070	14.30	1.002	0.998	86
5°	0.087	11.47	0.087	11.43	1.004	0.996	85°
6	0.105	9.567	0.105	9.514	1.006	0.995	84
7	0.122	8.206	0.123	8.144	1.008	0.993	83
8	0.139	7.185	0.141	7.115	1.010	0.990	82
9	0.156	6.392	0.158	6.314	1.012	0.988	81
10°	0.174	5.759	0.176	5.671	1.015	0.985	80°
11	0.191	5.241	0.194	5.145	1.019	0.982	79
12	0.208	4.810	0.213	4.705	1.022	0.978	78
13	0.225	4.445	0.231	4.331	1.026	0.974	77
14	0.242	4.134	0.249	4.011	1.031	0.970	76
15°	0.259	3.864	0.268	3.732	1.035	0.966	75°
16	0.276	3.628	0.287	3.487	1.040	0.961	74
17	0.292	3.420	0.306	3.271	1.046	0.956	73
18	0.309	3.236	0.325	3.078	1.051	0.951	72
19	0.326	3.072	0.344	2.904	1.058	0.946	71
20°	0.342	2.924	0.364	2.747	1.064	0.940	70°
21	0.358	2.790	0.384	2.605	1.071	0.934	69
22	0.375	2.669	0.404	2.475	1.079	0.927	68
23	0.391	2.559	0.424	2.356	1.086	0.921	67
24	0.407	2.459	0.445	2.246	1.095	0.914	66
25°	0.423	2.366	0.466	2.145	1.103	0.906	65°
26	0.438	2.281	0.488	2.050	1.113	0.899	64
27	0.454	2.203	0.510	1.963	1.122	0.891	63
28	0.469	2.130	0.532	1.881	1.133	0.883	62
29	0.485	2.063	0.554	1.804	1.143	0.875	61
30°	0.500	2.000	0.577	1.732	1.155	0.866	60°
31	0.515	1.942	0.601	1.664	1.167	0.857	59
32	0.530	1.887	0.625	1.600	1.179	0.848	58
33	0.545	1.836	0.649	1.540	1.192	0.839	57
34	0.559	1.788	0.675	1.483	1.206	0.829	56
35°	0.574	1.743	0.700	1.428	1.221	0.819	55°
36	0.588	1.701	0.727	1.376	1.236	0.809	54
37	0.602	1.662	0.754	1.327	1.252	0.799	53
38	0.616	1.624	0.781	1.280	1.269	0.788	52
39	0.629	1.589	0.810	1.235	1.287	0.777	51
40°	0.643	1.556	0.839	1.192	1.305	0.766	50°
41	0.656	1.524	0.869	1.150	1.325	0.755	49
42	0.669	1.494	0.900	1.111	1.346	0.743	48
43	0.682	1.466	0.933	1.072	1.367	0.731	47
44	0.695	1.440	0.966	1.036	1.390	0.719	46
45°	0.707	1.414	1.000	1.000	1.414	0.707	45°
	Cos.	Sec.	Ctn.	Tan.	Csc.	Sin.	Angle.

Logarithms.

N	0	1	2	3	4	5	6	7	8	9	P. P. 1. 2. 3. 4. 5
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4. 8.12.17.21
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 8.11.15.19
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3. 7.10.14.17
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3. 6.10.13.16
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3. 6. 9.12.15
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3. 6. 8.11.14
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3. 5. 8.11.13
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2. 5. 7.10.12
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2. 5. 7. 9.12
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2. 4. 7. 9.11
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2. 4. 6. 8.11
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2. 4. 6. 8.10
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2. 4. 6. 8.10
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2. 4. 5. 7. 9
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2. 4. 5. 7. 9
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2. 3. 5. 7. 9
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2. 3. 5. 7. 8
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2. 3. 5. 6. 8
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2. 3. 5. 6. 8
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1. 3. 4. 6. 7
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1. 3. 4. 6. 7
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1. 3. 4. 6. 7
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1. 3. 4. 5. 7
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1. 3. 4. 5. 6
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1. 3. 4. 5. 6
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1. 2. 4. 5. 6
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1. 2. 4. 5. 6
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1. 2. 3. 5. 6
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1. 2. 3. 5. 6
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1. 2. 3. 4. 6
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1. 2. 3. 4. 5
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1. 2. 3. 4. 5
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1. 2. 3. 4. 5
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1. 2. 3. 4. 5
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1. 2. 3. 4. 5
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1. 2. 3. 4. 5
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1. 2. 3. 4. 5
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1. 2. 3. 4. 5
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1. 2. 3. 4. 4
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1. 2. 3. 4. 4
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1. 2. 3. 3. 4
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1. 2. 3. 3. 4
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1. 2. 2. 3. 4
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1. 2. 2. 3. 4
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1. 2. 2. 3. 4

NOTE.—This page and the three that follow it are taken from the Mathematical Tables of Prof. J. M. Peirce, published by Messrs. Ginn & Co.

Logarithms.

N	0	1	2	3	4	5	6	7	8	9	P. P. 1.2.3.4.5
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1.2.2.3.4
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1.2.2 3.4
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1.2.2.3.4
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1.1.2.3.4
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1.1.2.3 4
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1.1.2.3.4
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1.1.2.3.4
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1.2.3.3
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1.1.2.3.3
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1.1.2.3.3
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1.1.2.3.3
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1.1.2.3.3
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1.1.2.3.3
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1.1.2.3.3
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1.1.2.3.3
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1.1.2.2.3
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1.1.2.2.3
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1.1.2.2.3
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1.1.2.2.3
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1.1.2.2.3
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1.1.2.2.3
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1.1.2.2.3
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1.1.2.2.3
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1.1.2.2.3
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1.1.2.2.3
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1.1.2.2.3
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1.1.2.2.3
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1.1.2.2.3
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1.1.2.2.3
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1.1.2.2.3
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1.1.2 2.3
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1.1.2.2.3
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0.1.1.2.2
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0.1.1.2.2
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0.1.1.2.2
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0.1.1.2.2
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0.1.1.2.2
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0.1.1.2.2
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0.1.1.2.2
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0.1.1.2.2
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0.1.1.2.2
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0.1.1.2.2
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0.1.1.2.2
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1.1.2.2
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0.1.1.2.2

$\log \pi = 0.49715 -.$ $\log e = 0.43429 +.$

Logarithms.

N	0	1	2	3	4	5	6	7	8	9	10
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039	0043
101	0043	0048	0052	0056	0060	0065	0069	0073	0077	0082	0086
102	0086	0090	0095	0099	0103	0107	0111	0116	0120	0124	0128
103	0128	0133	0137	0141	0145	0149	0154	0158	0162	0166	0170
104	0170	0175	0179	0183	0187	0191	0195	0199	0204	0208	0212
105	0212	0216	0220	0224	0228	0233	0237	0241	0245	0249	0253
106	0253	0257	0261	0265	0269	0273	0278	0282	0286	0290	0294
107	0294	0298	0302	0306	0310	0314	0318	0322	0326	0330	0334
108	0334	0338	0342	0346	0350	0354	0358	0362	0366	0370	0374
109	0374	0378	0382	0386	0390	0394	0398	0402	0406	0410	0414
110	0414	0418	0422	0426	0430	0434	0438	0441	0445	0449	0453
111	0453	0457	0461	0465	0469	0473	0477	0481	0484	0488	0492
112	0492	0496	0500	0504	0508	0512	0515	0519	0523	0527	0531
113	0531	0535	0538	0542	0546	0550	0554	0558	0561	0565	0569
114	0569	0573	0577	0580	0584	0588	0592	0596	0599	0603	0607
115	0607	0611	0615	0618	0622	0626	0630	0633	0637	0641	0645
116	0645	0648	0652	0656	0660	0663	0667	0671	0674	0678	0682
117	0682	0686	0689	0693	0697	0700	0704	0708	0711	0715	0719
118	0719	0722	0726	0730	0734	0737	0741	0745	0748	0752	0755
119	0755	0759	0763	0766	0770	0774	0777	0781	0785	0788	0792
120	0792	0795	0799	0803	0806	0810	0813	0817	0821	0824	0828
121	0828	0831	0835	0839	0842	0846	0849	0853	0856	0860	0864
122	0864	0867	0871	0874	0878	0881	0885	0888	0892	0896	0899
123	0899	0903	0906	0910	0913	0917	0920	0924	0927	0931	0934
124	0934	0938	0941	0945	0948	0952	0955	0959	0962	0966	0969
125	0969	0973	0976	0980	0983	0986	0990	0993	0997	1000	1004
126	1004	1007	1011	1014	1017	1021	1024	1028	1031	1035	1038
127	1038	1041	1045	1048	1052	1055	1059	1062	1065	1069	1072
128	1072	1075	1079	1082	1086	1089	1092	1096	1099	1103	1106
129	1106	1109	1113	1116	1119	1123	1126	1129	1133	1136	1139
130	1139	1143	1146	1149	1153	1156	1159	1163	1166	1169	1173
131	1173	1176	1179	1183	1186	1189	1193	1196	1199	1202	1206
132	1206	1209	1212	1216	1219	1222	1225	1229	1232	1235	1239
133	1239	1242	1245	1248	1252	1255	1258	1261	1265	1268	1271
134	1271	1274	1278	1281	1284	1287	1290	1294	1297	1300	1303
135	1303	1307	1310	1313	1316	1319	1323	1326	1329	1332	1335
136	1335	1339	1342	1345	1348	1351	1355	1358	1361	1364	1367
137	1367	1370	1374	1377	1380	1383	1386	1389	1392	1396	1399
138	1399	1402	1405	1408	1411	1414	1418	1421	1424	1427	1430
139	1430	1433	1436	1440	1443	1446	1449	1452	1455	1458	1461
140	1461	1464	1467	1471	1474	1477	1480	1483	1486	1489	1492
141	1492	1495	1498	1501	1504	1508	1511	1514	1517	1520	1523
142	1523	1526	1529	1532	1535	1538	1541	1544	1547	1550	1553
143	1553	1556	1559	1562	1565	1569	1572	1575	1578	1581	1584
144	1584	1587	1590	1593	1596	1599	1602	1605	1608	1611	1614
145	1614	1617	1620	1623	1626	1629	1632	1635	1638	1641	1644
146	1644	1647	1649	1652	1655	1658	1661	1664	1667	1670	1673
147	1673	1676	1679	1682	1685	1688	1691	1694	1697	1700	1703
148	1703	1706	1708	1711	1714	1717	1720	1723	1726	1729	1732
149	1732	1735	1738	1741	1744	1746	1749	1752	1755	1758	1761

Logarithms.

N	0	1	2	3	4	5	6	7	8	9	10
150	1761	1764	1767	1770	1772	1775	1778	1781	1784	1787	1790
151	1790	1793	1796	1798	1801	1804	1807	1810	1813	1816	1818
152	1818	1821	1824	1827	1830	1833	1836	1838	1841	1844	1847
153	1847	1850	1853	1855	1858	1861	1864	1867	1870	1872	1875
154	1875	1878	1881	1884	1886	1889	1892	1895	1898	1901	1903
155	1903	1906	1909	1912	1915	1917	1920	1923	1926	1928	1931
156	1931	1934	1937	1940	1942	1945	1948	1951	1953	1956	1959
157	1959	1962	1965	1967	1970	1973	1976	1978	1981	1984	1987
158	1987	1989	1992	1995	1998	2000	2003	2006	2009	2011	2014
159	2014	2017	2019	2022	2025	2028	2030	2033	2036	2038	2041
160	2041	2044	2047	2049	2052	2055	2057	2060	2063	2066	2068
161	2068	2071	2074	2076	2079	2082	2084	2087	2090	2092	2095
162	2095	2098	2101	2103	2106	2109	2111	2114	2117	2119	2122
163	2122	2125	2127	2130	2133	2135	2138	2140	2143	2146	2148
164	2148	2151	2154	2156	2159	2162	2164	2167	2170	2172	2175
165	2175	2177	2180	2183	2185	2188	2191	2193	2196	2198	2201
166	2201	2204	2206	2209	2212	2214	2217	2219	2222	2225	2227
167	2227	2230	2232	2235	2238	2240	2243	2245	2248	2251	2253
168	2253	2256	2258	2261	2263	2266	2269	2271	2274	2276	2279
169	2279	2281	2284	2287	2280	2292	2294	2297	2299	2302	2304
170	2304	2307	2310	2312	2315	2317	2320	2322	2325	2327	2330
171	2330	2333	2335	2338	2340	2343	2345	2348	2350	2353	2355
172	2355	2358	2360	2363	2365	2368	2370	2373	2375	2378	2380
173	2380	2383	2385	2388	2390	2393	2395	2398	2400	2403	2405
174	2405	2408	2410	2413	2415	2418	2420	2423	2425	2428	2430
175	2430	2433	2435	2438	2440	2443	2445	2448	2450	2453	2455
176	2455	2458	2460	2463	2465	2467	2470	2472	2475	2477	2480
177	2480	2482	2485	2487	2490	2492	2494	2497	2499	2502	2504
178	2504	2507	2509	2512	2514	2516	2519	2521	2524	2526	2529
179	2529	2531	2533	2536	2538	2541	2543	2545	2548	2550	2553
180	2553	2555	2558	2560	2562	2565	2567	2570	2572	2574	2577
181	2577	2579	2582	2584	2586	2589	2591	2594	2596	2598	2601
182	2601	2603	2605	2608	2610	2613	2615	2617	2620	2622	2625
183	2625	2627	2629	2632	2634	2636	2639	2641	2643	2646	2648
184	2648	2651	2653	2655	2658	2660	2662	2665	2667	2669	2672
185	2672	2674	2676	2679	2681	2683	2686	2688	2690	2693	2695
186	2695	2697	2700	2702	2704	2707	2709	2711	2714	2716	2718
187	2718	2721	2723	2725	2728	2730	2732	2735	2737	2739	2742
188	2742	2744	2746	2749	2751	2753	2755	2758	2760	2762	2765
189	2765	2767	2769	2772	2774	2776	2778	2781	2783	2785	2788
190	2788	2790	2792	2794	2797	2799	2801	2804	2806	2808	2810
191	2810	2813	2815	2817	2819	2822	2824	2826	2828	2831	2833
192	2833	2835	2838	2840	2842	2844	2847	2849	2851	2853	2856
193	2856	2858	2860	2862	2865	2867	2869	2871	2874	2876	2878
194	2878	2880	2882	2885	2887	2889	2891	2894	2896	2898	2900
195	2900	2903	2905	2907	2909	2911	2914	2916	2918	2920	2923
196	2923	2925	2927	2929	2931	2934	2936	2938	2940	2942	2945
197	2945	2947	2949	2951	2953	2956	2958	2960	2962	2964	2967
198	2967	2969	2971	2973	2975	2978	2980	2982	2984	2986	2989
199	2989	2991	2993	2995	2997	2999	3002	3004	3006	3008	3010

Trigonometric Functions.

RADIANS.	DEGREES.	SINES.		COSINES.		TANGENTS.		COTANGENTS.			
		Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.		
0.0000	0° 00′	.0000	∞	1.0000	0.0000	.0000	∞	∞	∞	90° 00′	1.5708
0.0029	10	.0029	7.4637	1.0000	.0000	.0029	7.4637	343.77	2.5363	50	1.5679
0.0058	20	.0058	.7648	1.0000	.0000	.0058	.7648	171.89	.2352	40	1.5650
0.0087	30	.0087	.9408	1.0000	.0000	.0087	.9409	114.59	.0591	30	1.5621
0.0116	40	.0116	8.0658	.9999	.0000	.0116	8.0658	85.940	1.9342	20	1.5592
0.0145	50	.0145	.1627	.9999	.0000	.0145	.1627	68.750	.8373	10	1.5563
0.0175	1° 00′	.0175	8.2419	.9998	9.9999	.0175	8.2419	57.290	1.7581	89° 00′	1.5533
0.0204	10	.0204	.3088	.9998	.9999	.0204	.3089	49.104	.6911	50	1.5504
0.0233	20	.0233	.3668	.9997	.9999	.0233	.3669	42.964	.6331	40	1.5475
0.0262	30	.0262	.4179	.9997	.9999	.0262	.4181	38.188	.5819	30	1.5446
0.0291	40	.0291	.4637	.9996	.9998	.0291	.4638	34.368	.5362	20	1.5417
0.0320	50	.0320	.5050	.9995	.9998	.0320	.5053	31.242	.4947	10	1.5388
0.0349	2° 00′	.0349	8.5428	.9994	9.9997	.0349	8.5431	28.636	1.4569	88° 00′	1.5359
0.0378	10	.0378	.5776	.9993	.9997	.0378	.5779	26.432	.4221	50	1.5330
0.0407	20	.0407	.6097	.9992	.9996	.0407	.6101	24.542	.3899	40	1.5301
0.0436	30	.0436	.6397	.9990	.9996	.0437	.6401	22.904	.3599	30	1.5272
0.0465	40	.0465	.6677	.9989	.9995	.0466	.6682	21.470	.3318	20	1.5243
0.0495	50	.0494	.6940	.9988	.9995	.0495	.6945	20.206	.3055	10	1.5213
0.0524	3° 00′	.0523	8.7188	.9986	9.9994	.0524	8.7194	19.081	1.2806	87° 00′	1.5184
0.0553	10	.0552	.7423	.9985	.9993	.0553	.7429	18.075	.2571	50	1.5155
0.0582	20	.0581	.7645	.9983	.9993	.0582	.7652	17.169	.2348	40	1.5126
0.0611	30	.0610	.7857	.9981	.9992	.0612	.7865	16.350	.2135	30	1.5097
0.0640	40	.0640	.8059	.9980	.9991	.0641	.8067	15.605	.1933	20	1.5068
0.0669	50	.0669	.8251	.9978	.9990	.0670	.8261	14.924	.1739	10	1.5039
0.0698	4° 00′	.0698	8.8436	.9976	9.9989	.0699	8.8446	14.301	1.1554	86° 00′	1.5010
0.0727	10	.0727	.8613	.9974	.9989	.0729	.8624	13.727	.1376	50	1.4981
0.0756	20	.0756	.8783	.9971	.9988	.0758	.8795	13.197	.1205	40	1.4952
0.0785	30	.0785	.8946	.9969	.9987	.0787	.8960	12.706	.1040	30	1.4923
0.0814	40	.0814	.9104	.9967	.9986	.0816	.9118	12.251	.0882	20	1.4893
0.0844	50	.0843	.9256	.9964	.9985	.0846	.9272	11.826	.0728	10	1.4864
0.0873	5° 00′	.0872	8.9403	.9962	9.9983	.0875	8.9420	11.430	1.0580	85° 00′	1.4835
0.0902	10	.0901	.9545	.9959	.9982	.0904	.9563	11.059	.0437	50	1.4806
0.0931	20	.0929	.9682	.9957	.9981	.0934	.9701	10.712	.0299	40	1.4777
0.0960	30	.0958	.9816	.9954	.9980	.0963	.9836	10.385	.0164	30	1.4748
0.0989	40	.0987	.9945	.9951	.9979	.0992	.9966	10.078	.0034	20	1.4719
0.1018	50	.1016	9.0070	.9948	.9977	.1022	9.0093	9.7882	0.9907	10	1.4690
0.1047	6° 00′	.1045	9.0192	.9945	9.9976	.1051	9.0216	9.5144	0.9784	84° 00′	1.4661
0.1076	10	.1074	.0311	.9942	.9975	.1080	.0336	9.2553	.9664	50	1.4632
0.1105	20	.1103	.0426	.9939	.9973	.1110	.0453	9.0098	.9547	40	1.4603
0.1134	30	.1132	.0539	.9936	.9972	.1139	.0567	8.7769	.9433	30	1.4574
0.1164	40	.1161	.0648	.9932	.9971	.1169	.0678	8.5555	.9322	20	1.4544
0.1193	50	.1190	.0755	.9929	.9969	.1198	.0786	8.3450	.9214	10	1.4515
0.1222	7° 00′	.1219	9.0859	.9925	9.9968	.1228	9.0891	8.1443	0.9109	83° 00′	1.4486
0.1251	10	.1248	.0961	.9922	.9966	.1257	.0995	7.9530	.9005	50	1.4457
0.1280	20	.1276	.1060	.9918	.9964	.1287	.1096	7.7704	.8904	40	1.4428
0.1309	30	.1305	.1157	.9914	.9963	.1317	.1194	7.5958	.8806	30	1.4399
0.1338	40	.1334	.1252	.9911	.9961	.1346	.1291	7.4287	.8709	20	1.4370
0.1367	50	.1363	.1345	.9907	.9959	.1376	.1385	7.2687	.8615	10	1.4341
0.1396	8° 00′	.1392	9.1436	.9903	9.9958	.1405	9.1478	7.1154	0.8522	82° 00′	1.4312
0.1425	10	.1421	.1525	.9899	.9956	.1435	.1569	6.9682	.8431	50	1.4283
0.1454	20	.1449	.1612	.9894	.9954	.1465	.1658	6.8269	.8342	40	1.4254
0.1484	30	.1478	.1697	.9890	.9952	.1495	.1745	6.6912	.8255	30	1.4224
0.1513	40	.1507	.1781	.9886	.9950	.1524	.1831	6.5606	.8169	20	1.4195
0.1542	50	.1536	.1863	.9881	.9948	.1554	.1915	6.4348	.8085	10	1.4166
0.1571	9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	81° 00′	1.4137
		Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.		
		COSINES.		SINES.		COTANGENTS.		TANGENTS.		DEGREES.	RADIANS.

Trigonometric Functions.

RADIANS.	DEGREES.	SINES.		COSINES.		TANGENTS.		COTANGENTS.			
		Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.		
0.1571	9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	81° 00′	1.4137
0.1600	10	.1593	.2022	.9872	.9944	.1614	.2078	6.1970	.7922	50	1.4108
0.1629	20	.1622	.2100	.9868	.9942	.1644	.2158	6.0844	.7842	40	1.4079
0.1658	30	.1650	.2176	.9863	.9940	.1673	.2236	5.9758	.7764	30	1.4050
0.1687	40	.1679	.2251	.9858	.9938	.1703	.2313	5.8708	.7687	20	1.4021
0.1716	50	.1708	.2324	.9853	.9936	.1733	.2389	5.7694	.7611	10	1.3992
0.1745	10° 00′	.1736	9.2397	.9848	9.9934	.1763	9.2463	5.6713	0.7537	80° 00′	1.3963
0.1774	10	.1765	.2468	.9843	.9931	.1793	.2536	5.5764	.7464	50	1.3934
0.1804	20	.1794	.2538	.9838	.9929	.1823	.2609	5.4845	.7391	40	1.3904
0.1833	30	.1822	.2606	.9833	.9927	.1853	.2680	5.3955	.7320	30	1.3875
0.1862	40	.1851	.2674	.9827	.9924	.1883	.2750	5.3093	.7250	20	1.3846
0.1891	50	.1880	.2740	.9822	.9922	.1914	.2819	5.2257	.7181	10	1.3817
0.1920	11° 00′	.1908	9.2806	.9816	9.9919	.1944	9.2887	5.1446	0.7113	79° 00′	1.3788
0.1949	10	.1937	.2870	.9811	.9917	.1974	.2953	5.0658	.7047	50	1.3759
0.1978	20	.1965	.2934	.9805	.9914	.2004	.3020	4.9894	.6980	40	1.3730
0.2007	30	.1994	.2997	.9799	.9912	.2035	.3085	4.9152	.6915	30	1.3701
0.2036	40	.2022	.3058	.9793	.9909	.2065	.3149	4.8430	.6851	20	1.3672
0.2065	50	.2051	.3119	.9787	.9907	.2095	.3212	4.7729	.6788	10	1.3643
0.2094	12° 00′	.2079	9.3179	.9781	9.9904	.2126	9.3275	4.7046	0.6725	78° 00′	1.3614
0.2123	10	.2108	.3238	.9775	.9901	.2156	.3336	4.6382	.6664	50	1.3584
0.2153	20	.2136	.3296	.9769	.9899	.2186	.3397	4.5736	.6603	40	1.3555
0.2182	30	.2164	.3353	.9763	.9896	.2217	.3458	4.5107	.6542	30	1.3526
0.2211	40	.2193	.3410	.9757	.9893	.2247	.3517	4.4494	.6483	20	1.3497
0.2240	50	.2221	.3466	.9750	.9890	.2278	.3576	4.3897	.6424	10	1.3468
0.2269	13° 00′	.2250	9.3521	.9744	9.9887	.2309	9.3634	4.3315	0.6366	77° 00′	1.3439
0.2298	10	.2278	.3575	.9737	.9884	.2339	.3691	4.2747	.6309	50	1.3410
0.2327	20	.2306	.3629	.9730	.9881	.2370	.3748	4.2193	.6252	40	1.3381
0.2356	30	.2334	.3682	.9724	.9878	.2401	.3804	4.1653	.6196	30	1.3352
0.2385	40	.2363	.3734	.9717	.9875	.2432	.3859	4.1126	.6141	20	1.3323
0.2414	50	.2391	.3786	.9710	.9872	.2462	.3914	4.0611	.6086	10	1.3294
0.2443	14° 00′	.2419	9.3837	.9703	9.9869	.2493	9.3968	4.0108	0.6032	76° 00′	1.3265
0.2473	10	.2447	.3887	.9696	.9866	.2524	.4021	3.9617	.5979	50	1.3235
0.2502	20	.2476	.3937	.9689	.9863	.2555	.4074	3.9136	.5926	40	1.3206
0.2531	30	.2504	.3986	.9681	.9859	.2586	.4127	3.8667	.5873	30	1.3177
0.2560	40	.2532	.4035	.9674	.9856	.2617	.4178	3.8208	.5822	20	1.3148
0.2589	50	.2560	.4083	.9667	.9853	.2648	.4230	3.7760	.5770	10	1.3119
0.2618	15° 00′	.2588	9.4130	.9659	9.9849	.2679	9.4281	3.7321	0.5719	75° 00′	1.3090
0.2647	10	.2616	.4177	.9652	.9846	.2711	.4331	3.6891	.5669	50	1.3061
0.2676	20	.2644	.4223	.9644	.9843	.2742	.4381	3.6470	.5619	40	1.3032
0.2705	30	.2672	.4269	.9636	.9839	.2773	.4430	3.6059	.5570	30	1.3003
0.2734	40	.2700	.4314	.9628	.9836	.2805	.4479	3.5656	.5521	20	1.2974
0.2763	50	.2728	.4359	.9621	.9832	.2836	.4527	3.5261	.5473	10	1.2945
0.2793	16° 00′	.2756	9.4403	.9613	9.9828	.2867	9.4575	3.4874	0.5425	74° 00′	1.2915
0.2822	10	.2784	.4447	.9605	.9825	.2899	.4622	3.4495	.5378	50	1.2886
0.2851	20	.2812	.4491	.9596	.9821	.2931	.4669	3.4124	.5331	40	1.2857
0.2880	30	.2840	.4533	.9588	.9817	.2962	.4716	3.3759	.5284	30	1.2828
0.2909	40	.2868	.4576	.9580	.9814	.2994	.4762	3.3402	.5238	20	1.2799
0.2938	50	.2896	.4618	.9572	.9810	.3026	.4808	3.3052	.5192	10	1.2770
0.2967	17° 00′	.2924	9.4659	.9563	9.9806	.3057	9.4853	3.2709	0.5147	73° 00′	1.2741
0.2996	10	.2952	.4700	.9555	.9802	.3089	.4898	3.2371	.5102	50	1.2712
0.3025	20	.2979	.4741	.9546	.9798	.3121	.4943	3.2041	.5057	40	1.2683
0.3054	30	.3007	.4781	.9537	.9794	.3153	.4987	3.1716	.5013	30	1.2654
0.3083	40	.3035	.4821	.9528	.9790	.3185	.5031	3.1397	.4969	20	1.2625
0.3113	50	.3062	.4861	.9520	.9786	.3217	.5075	3.1084	.4925	10	1.2595
0.3142	18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	72° 00′	1.2566
		Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.		
		COSINES.		SINES.		COTANGENTS.		TANGENTS.		DEGREES.	RADIANS.

Trigonometric Functions.

RADIANS.	DEGREES.	SINES.		COSINES.		TANGENTS.		COTANGENTS.			
		Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.		
0.3142	18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	72° 00′	1.2566
0.3171	10	.3118	.4939	.9502	.9778	.3281	.5161	3.0475	.4839	50	1.2537
0.3200	20	.3145	.4977	.9492	.9774	.3314	.5203	3.0178	.4797	40	1.2508
0.3229	30	.3173	.5015	.9483	.9770	.3346	.5245	2.9887	.4755	30	1.2479
0.3258	40	.3201	.5052	.9474	.9765	.3378	.5287	2.9600	.4713	20	1.2450
0.3287	50	.3228	.5090	.9465	.9761	.3411	.5329	2.9319	.4671	10	1.2421
0.3316	19° 00′	.3256	9.5126	.9455	9.9757	.3443	9.5370	2.9042	0.4630	71° 00′	1.2392
0.3345	10	.3283	.5163	.9446	.9752	.3476	.5411	2.8770	.4589	50	1.2363
0.3374	20	.3311	.5199	.9436	.9748	.3408	.5451	2.8502	.4549	40	1.2334
0.3403	30	.3338	.5235	.9426	.9743	.3541	.5491	2.8239	.4509	30	1.2305
0.3432	40	.3365	.5270	.9417	.9739	.3574	.5531	2.7980	.4469	20	1.2275
0.3462	50	.3393	.5306	.9407	.9734	.3607	.5571	2.7725	.4429	10	1.2246
0.3491	20° 00′	.3420	9.5341	.9397	9.9730	.3640	9.5611	2.7475	0.4389	70° 00′	1.2217
0.3520	10	.3448	.5375	.9387	.9725	.3673	.5650	2.7228	.4350	50	1.2188
0.3549	20	.3475	.5409	.9377	.9721	.3706	.5689	2.6985	.4311	40	1.2159
0.3578	30	.3502	.5443	.9367	.9716	.3739	.5727	2.6746	.4273	30	1.2130
0.3607	40	.3529	.5477	.9356	.9711	.3772	.5766	2.6511	.4234	20	1.2101
0.3636	50	.3557	.5510	.9346	.9706	.3805	.5804	2.6279	.4196	10	1.2072
0.3665	21° 00′	.3584	9.5543	.9336	9.9702	.3839	9.5842	2 6051	0.4158	69° 00′	1.2043
0.3694	10	.3611	.5576	.9325	.9697	.3872	.5879	2.5826	.4121	50	1.2014
0.3723	20	.3638	.5609	.9315	.9692	.3906	.5917	2.5605	.4083	40	1.1985
0.3752	30	.3665	.5641	.9304	.9687	.3939	.5954	2 5386	.4046	30	1.1956
0.3782	40	.3692	.5673	.9293	.9682	.3973	.5991	2.5172	.4009	20	1.1926
0.3811	50	.3719	.5704	.9283	.9677	.4006	.6028	2.4960	.3972	10	1.1897
0.3840	22° 00′	.3746	9.5736	.9272	9.9672	.4040	9.6064	2.4751	0.3936	68° 00′	1.1868
0.3869	10	.3773	.5767	.9261	.9667	.4074	.6100	2.4545	.3900	50	1.1839
0.3898	20	.3800	.5798	.9250	.9661	.4108	.6136	2.4342	.3864	40	1.1810
0.3927	30	.3827	.5828	.9239	.9656	.4142	.6172	2.4142	.3828	30	1.1781
0.3956	40	.3854	.5859	.9228	.9651	.4176	.6208	2.3945	.3792	20	1.1752
0.3985	50	.3881	.5889	.9216	.9646	.4210	.6243	2 3750	.3757	10	1.1723
0.4014	23° 00′	.3907	9.5919	.9205	9.9640	.4245	9.6279	2.3559	0.3721	67° 00′	1.1694
0.4043	10	.3934	.5948	.9194	.9635	.4279	.6314	2 3369	.3686	50	1.1665
0.4072	20	.3961	.5978	.9182	.9629	.4314	.6348	2.3183	.3652	40	1.1636
0.4102	30	.3987	.6007	.9171	.9624	.4348	.6383	2.2998	.3617	30	1.1606
0.4131	40	.4014	.6036	.9159	.9618	.4383	.6417	2 2817	.3583	20	1.1577
0.4160	50	.4041	.6065	.9147	.9613	.4417	.6452	2.2637	.3548	10	1.1548
0.4189	24° 00′	.4067	9.6093	.9135	9.9607	.4452	9.6486	2.2460	0.3514	66° 00′	1.1519
0.4218	10	.4094	.6121	.9124	.9602	.4487	.6520	2.2286	.3480	50	1.1490
0.4247	20	.4120	.6149	.9112	.9596	.4522	.6553	2.2113	.3447	40	1.1461
0.4276	30	.4147	.6177	.9100	.9590	.4557	.6587	2.1943	.3413	30	1.1432
0.4305	40	.4173	.6205	.9088	.9584	.4592	.6620	2.1775	.3380	20	1 1403
0.4334	50	.4100	.6232	.9075	.9579	.4628	.6654	2.1609	.3346	10	1.1374
0.4363	25° 00′	.4226	9.6259	.9063	9.9573	.4663	9.6687	2.1445	0.3313	65° 00′	1.1345
0.4392	10	.4253	.6286	.9051	.9567	.4699	.6720	2.1283	.3280	50	1.1316
0.4422	20	.4279	.6313	.9038	.9561	.4734	.6752	2.1123	.3248	40	1.1286
0.4451	30	.4305	.6340	.9026	.9555	.4770	.6785	2.0965	.3215	30	1.1257
0.4480	40	.4331	.6366	.9013	.9549	.4806	.6817	2.0809	.3183	20	1.1228
0.4509	50	.4358	.6392	.9001	.9543	.4841	.6850	2.0655	.3150	10	1.1199
0.4538	26° 00′	.4384	9.6418	.8988	9.9537	.4877	9.6882	2.0503	0.3118	64° 00′	1.1170
0.4567	10	.4410	.6444	.8975	.9530	.4913	.6914	2.0353	.3086	50	1.1141
0.4596	20	.4436	.6470	.8962	.9524	.4950	.6946	2.0204	.3054	40	1.1112
0.4625	30	.4462	.6495	.8949	.9518	.4986	.6977	2.0057	.3023	30	1.1083
0.4654	40	.4488	.6521	.8936	.9512	.5022	.7009	1.9912	.2991	20	1.1054
0.4683	50	.4514	.6546	.8923	.9505	.5059	.7040	1.9768	.2960	10	1.1025
0.4712	27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	63° 00′	1.0996
		Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.		
		COSINES.		SINES.		COTANGENTS.		TANGENTS.		DEGREES.	RADIANS.

Trigonometric Functions.

RADIANS.	DEGREES.	SINES.		COSINES.		TANGENTS.		COTANGENTS.			
		Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.		
0.4712	27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	63° 00′	1.0996
0.4741	10	.4566	.6595	.8897	.9492	.5132	.7103	1.9486	.2897	50	1.0966
0.4771	20	.4592	.6620	.8884	.9486	.5169	.7134	1.9347	.2866	40	1.0937
0.4800	30	.4617	.6644	.8870	.9479	.5206	,7165	1.9210	.2835	30	1.0908
0.4829	40	.4643	.6668	.8857	.9473	.5243	.7196	1.9074	.2804	20	1.0879
0.4858	50	.4669	.6692	.8843	.9466	.5280	.7226	1.8940	.2774	10	1.0850
0.4887	28° 00′	.4695	9.6716	.8829	9.9459	.5317	9.7257	1.8807	0.2743	62° 00′	1.0821
0.4916	10	.4720	.6740	.8816	.9453	.5354	.7287	1.8676	.2713	50	1.0792
0.4945	20	.4746	.6763	.8802	.9446	.5392	.7317	1.8546	.2683	40	1.0763
0.4974	30	.4772	.6787	.8788	.9439	.5430	.7348	1.8418	.2652	30	1.0734
0.5003	40	.4797	.6810	.8774	.9432	.5467	.7378	1.8291	.2622	20	1.0705
0.5032	50	.4823	.6833	.8760	.9425	.5505	.7408	1.8165	.2592	10	1.0676
0.5061	29° 00′	.4848	9.6856	.8746	9.9418	.5543	9.7438	1.8040	0.2562	61° 00′	1.0647
0.5091	10	.4874	.6878	.8732	.9411	.5581	.7467	1.7917	.2533	50	1.0617
0.5120	20	.4899	.6901	.8718	.9404	.5619	.7497	1.7796	.2503	40	1.0588
0.5149	30	.4924	.6923	.8704	.9397	.5658	.7526	1.7675	.2474	30	1.0559
0.5178	40	.4950	.6946	.8689	.9390	.5696	.7556	1.7556	.2444	20	1.0530
0.5207	50	.4975	.6968	.8675	.9383	.5735	.7585	1.7437	.2415	10	1.0501
0.5236	30° 00′	.5000	9.6990	.8660	9.9375	.5774	9.7614	1.7321	0.2386	60° 00′	1.0472
0.5265	10	.5025	.7012	.8646	.9368	.5812	.7644	1.7205	.2356	50	1.0443
0.5294	20	.5050	.7033	.8631	.9361	.5851	.7673	1.7090	.2327	40	1.0414
0.5323	30	.5075	.7055	.8616	.9353	.5890	.7701	1.6977	.2299	30	1.0385
0.5352	40	.5100	.7076	.8601	.9346	.5930	.7730	1.6864	.2270	20	1.0356
0.5381	50	.5125	.7097	.8587	.9338	.5969	.7759	1.6753	.2241	10	1.0327
0.5411	31° 00′	.5150	9.7118	.8572	9.9331	.6009	9.7788	1.6643	0.2212	59° 00′	1.0297
0.5440	10	.5175	.7139	.8557	.9323	.6048	.7816	1.6534	.2184	50	1.0268
0.5469	20	.5200	.7160	.8542	.9315	.6088	.7845	1.6426	.2155	40	1.0239
0.5498	30	.5225	.7181	.8526	.9308	.6128	.7873	1.6319	.2127	30	1.0210
0.5527	40	.5250	.7201	.8511	.9300	.6168	.7902	1.6212	.2098	20	1.0181
0.5556	50	.5275	.7222	.8496	.9292	.6208	.7930	1.6107	.2070	10	1.0152
0.5585	32° 00′	.5299	9.7242	.8480	9.9284	.6249	9.7958	1.6003	0.2042	58° 00′	1.0123
0.5614	10	.5324	.7262	.8465	.9276	.6289	.7986	1.5900	.2014	50	1.0094
0.5643	20	.5348	.7282	.8450	.9268	.6330	.8014	1.5798	.1986	40	1.0065
0.5672	30	.5373	.7302	.8434	.9260	.6371	.8042	1.5697	.1958	30	1.0036
0.5701	40	.5398	.7322	.8418	.9252	.6412	.8070	1.5597	.1930	20	1.0007
0.5730	50	.5422	.7342	.8403	.9244	.6453	.8097	1.5497	.1903	10	0.9977
0.5760	33° 00′	.5446	9.7361	.8387	9.9236	.6494	9.8125	1.5399	0.1875	57° 00′	0.9948
0.5789	10	.5471	.7380	.8371	.9228	.6536	.8153	1.5301	.1847	50	0.9919
0.5818	20	.5495	.7400	.8355	.9219	.6577	.8180	1.5204	.1820	40	0.9890
0.5847	30	.5519	.7419	.8339	.9211	.6619	.8208	1.5108	.1792	30	0.9861
0.5876	40	.5544	.7438	.8323	.9203	.6661	.8235	1.5013	.1765	20	0.9832
0.5905	50	.5568	.7457	.8307	.9194	.6703	.8263	1.4919	.1737	10	0.9803
0.5934	34° 00′	.5592	9.7476	.8290	9.9186	.6745	9.8290	1.4826	0.1710	56° 00′	0.9774
0.5963	10	.5616	.7494	.8274	.9177	.6787	.8317	1.4733	.1683	50	0.9745
0.5992	20	.5640	.7513	.8258	.9169	.6830	.8344	1.4641	.1656	40	0.9716
0.6021	30	.5664	.7531	.8241	.9160	.6873	.8371	1.4550	.1629	30	0.9687
0.6050	40	.5688	.7550	.8225	.9151	.6916	.8398	1.4460	.1602	20	0.9657
0.6080	50	.5712	.7568	.8208	.9142	.6959	.8425	1.4370	.1575	10	0.9628
0.6109	35° 00′	.5736	9.7586	.8192	9.9134	.7002	9.8452	1.4281	0.1548	55° 00′	0.9599
0.6138	10	.5760	.7604	.8175	.9125	.7046	.8479	1.4193	.1521	50	0.9570
0.6167	20	.5783	.7622	.8158	.9116	.7089	.8506	1.4106	.1494	40	0.9541
0.6196	30	.5807	.7640	.8141	.9107	.7133	.8533	1.4019	.1467	30	0.9512
0.6225	40	.5831	.7657	.8124	.9098	.7177	.8559	1.3934	.1441	20	0.9483
0.6254	50	.5854	7675	.8107	.9089	.7221	.8586	1.3848	.1414	10	0.9454
0.6283	36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	54° 00′	0.9425
		Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.		
		COSINES.		SINES.		COTANGENTS.		TANGENTS.		DEGREES.	RADIANS.

Trigonometric Functions.

RADIANS.	DEGREES.	SINES.		COSINES.		TANGENTS.		COTANGENTS.			
		Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.		
0.6283	36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	54° 00′	0.9425
0.6312	10	.5901	.7710	.8073	.9070	.7310	.8639	1.3680	.1361	50	0.9396
0.6341	20	.5925	.7727	.8056	.9061	.7355	.8666	1.3597	.1334	40	0.9367
0.6370	30	.5948	.7744	.8039	.9052	.7400	.8692	1.3514	.1308	30	0.9338
0.6400	40	.5972	.7761	.8021	.9042	.7445	.8718	1.3432	.1282	20	0.9308
0.6429	50	.5995	.7778	.8004	.9033	.7490	.8745	1.3351	.1255	10	0.9279
0.6458	37° 00′	.6018	9.7795	.7986	9.9023	.7536	9.8771	1.3270	0.1229	53° 00′	0.9250
0.6487	10	.6041	.7811	.7969	.9014	.7581	.8797	1.3190	.1203	50	0.9221
0.6516	20	.6065	.7828	.7951	.9004	.7627	.8824	1.3111	.1176	40	0.9192
0.6545	30	.6088	.7844	.7934	.8995	.7673	.8850	1.3032	.1150	30	0.9163
0.6574	40	.6111	.7861	.7916	.8985	.7720	.8876	1.2954	.1124	20	0.9134
0.6603	50	.6134	.7877	.7898	.8975	.7766	.8902	1.2876	.1098	10	0.9105
0.6632	38° 00′	.6157	9.7893	.7880	9.8965	.7813	9.8928	1.2799	0.1072	52° 00′	0.9076
0.6661	10	.6180	.7910	.7862	.8955	.7860	.8954	1.2723	.1046	50	0.9047
0.6690	20	.6202	.7926	.7844	.8945	.7907	.8980	1.2647	.1020	40	0.9018
0.6720	30	.6225	.7941	.7826	.8935	.7954	.9006	1.2572	.0994	30	0.8988
0.6749	40	.6248	.7957	.7808	.8925	.8002	.9032	1.2497	.0968	20	0.8959
0.6778	50	.6271	.7973	.7790	.8915	.8050	.9058	1.2423	.0942	10	0.8930
0.6807	39° 00′	.6293	9.7989	.7771	9.8905	.8098	9.9084	1.2349	0.0916	51° 00′	0.8901
0.6836	10	.6316	.8004	.7753	.8895	.8146	.9110	1.2276	.0890	50	0.8872
0.6865	20	.6338	.8020	.7735	.8884	.8195	.9135	1.2203	.0865	40	0.8843
0.6894	30	.6361	.8035	.7716	.8874	.8243	.9161	1.2131	.0839	30	0.8814
0.6923	40	.6383	.8050	.7698	.8864	.8292	.9187	1.2059	.0813	20	0.8785
0.6952	50	.6406	.8066	.7679	.8853	.8342	.9212	1.1988	.0788	10	0.8756
0.6981	40° 00′	.6428	9.8081	.7660	9.8843	.8391	9.9238	1.1918	0.0762	50° 00′	0.8727
0.7010	10	.6450	.8096	.7642	.8832	.8441	.9264	1.1847	.0736	50	0.8698
0.7039	20	.6472	.8111	.7623	.8821	.8491	.9289	1.1778	.0711	40	0.8668
0.7069	30	.6494	.8125	.7604	.8810	.8541	.9315	1.1708	.0685	30	0.8639
0.7098	40	.6517	.8140	.7585	.8800	.8591	.9341	1.1640	.0659	20	0.8610
0.7127	50	.6539	.8155	.7566	.8789	.8642	.9366	1.1571	.0634	10	0.8581
0.7156	41° 00′	.6561	9.8169	.7547	9.8778	.8693	9.9392	1.1504	0.0608	49° 00′	0.8552
0.7185	10	.6583	.8184	.7528	.8767	.8744	.9417	1.1436	.0583	50	0.8523
0.7214	20	.6604	.8198	.7509	.8756	.8796	.9443	1.1369	.0557	40	0.8494
0.7243	30	.6626	.8213	.7490	.8745	.8847	.9468	1.1303	.0532	30	0.8465
0.7272	40	.6648	.8227	.7470	.8733	.8899	.9494	1.1237	.0506	20	0.8436
0.7301	50	.6670	.8241	.7451	.8722	.8952	.9519	1.1171	.0481	10	0.8407
0.7330	42° 00′	.6691	9.8255	.7431	9.8711	.9004	9.9544	1.1106	0.0456	48° 00′	0.8378
0.7359	10	.6713	.8269	.7412	.8699	.9057	.9570	1.1041	.0430	50	0.8348
0.7389	20	.6734	.8283	.7392	.8688	.9110	.9595	1.0977	.0405	40	0.8319
0.7418	30	.6756	.8297	.7373	.8676	.9163	.9621	1.0913	.0379	30	0.8290
0.7447	40	.6777	.8311	.7353	.8665	.9217	.9646	1.0850	.0354	20	0.8261
0.7476	50	.6799	.8324	.7333	.8653	.9271	.9671	1.0786	.0329	10	0.8232
0.7505	43° 00′	.6820	9.8338	.7314	9.8641	.9325	9.9697	1.0724	0.0303	47° 00′	0.8203
0.7534	10	.6841	.8351	.7294	.8629	.9380	.9722	1.0661	.0278	50	0.8174
0.7563	20	.6862	.8365	.7274	.8618	.9435	.9747	1.0599	.0253	40	0.8145
0.7592	30	.6884	.8378	.7254	.8606	.9490	.9772	1.0538	.0228	30	0.8116
0.7621	40	.6905	.8391	.7234	.8594	.9545	.9798	1.0477	.0202	20	0.8087
0.7650	50	.6926	.8405	.7214	.8582	.9601	.9823	1.0416	.0177	10	0.8058
0.7679	44° 00′	.6947	9.8418	.7193	9.8569	.9657	9.9848	1.0355	0.0152	46° 00′	0.8029
0.7709	10	.6967	.8431	.7173	.8557	.9713	.9874	1.0295	.0126	50	0.7999
0.7738	20	.6988	.8444	.7153	.8545	.9770	.9899	1.0235	.0101	40	0.7970
0.7767	30	.7009	.8457	.7133	.8532	.9827	.9924	1.0176	.0076	30	0.7941
0.7796	40	.7030	.8469	.7112	.8520	.9884	.9949	1.0117	.0051	20	0.7912
0.7825	50	.7050	.8482	.7092	.8507	.9942	.9975	1.0058	.0025	10	0.7883
0.7854	45° 00′	.7071	9.8495	.7071	9.8495	1.0000	0.0000	1.0000	0.0000	45° 00′	0.7854
		Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.		
		COSINES		SINES.		COTANGENTS.		TANGENTS.		DEGREES.	RADIANS.

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